Path integrals for de Sitter and anti-de Sitter space

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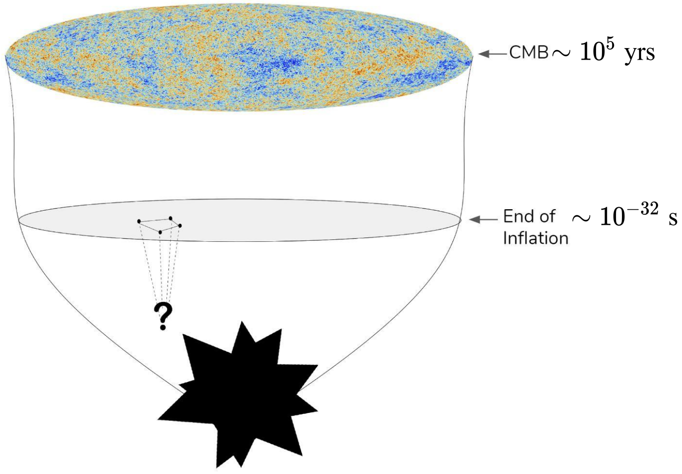
MAX-PLANCK-GESELLSCHAFT

- How do we understand gravity path integrals beyond the saddle point approximation?
- Boundary conditions and integration contours for path integrals in cosmology and holography
- Minisuperspace approximation
- No-boundary proposal
- Black holes in EAdS

based on 2007.04872 [hep-th]
Work in collaboration with Michal P. Heller and Jean-Luc Lehners

The wavefunction of the universe Ψ defines probabilities for fields configurations on a spacelike surface

wavefunction of the universe ⟶ cosmological correlators



$$\langle \prod_j^n f(\vec{p}_j) \rangle = \int \mathcal{D}f \prod_j^n f(\vec{p}_j) |\Psi(f)|^2$$

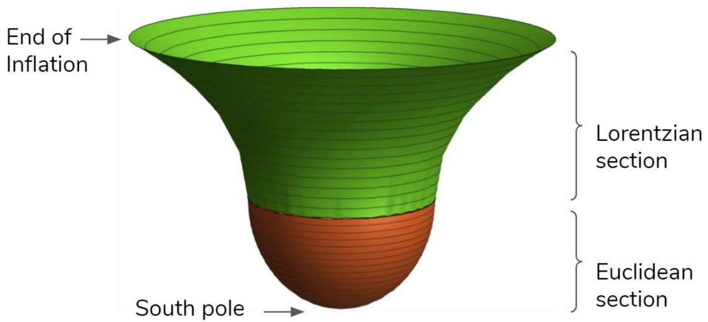
$$\Psi[f] = \int_{i.c.}^{f} \mathcal{D}\phi \, e^{iS(\phi)} \sim \exp[-\tfrac{1}{2} f f \psi_2 - \tfrac{1}{6} f f f \psi_3 + \cdots]$$

The result depends on the choice of initial conditions!

Bunch-Davies initial conditions = regularity condition on the HH geometry

No boundary proposal [Hartle Hawking ‘83]

$$\Psi_{HH} \approx$$



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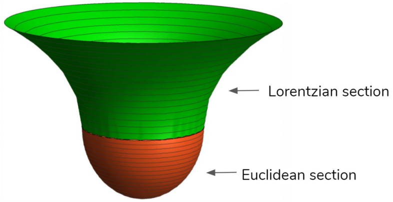
$$\Psi_{HH} = \int_{?} \delta g_{\mu\nu}\, e^{iS} \approx$$



How is the gravitational path integral actually defined?

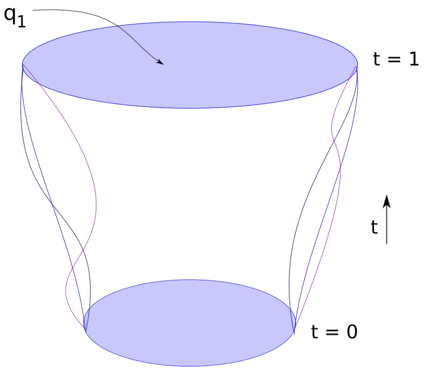
"The boundary condition of the universe is that it has no boundary"

''Our proposal is that the sum should be over compact geometries. This means that the Universe does not have any boundary in space or time. ''

The wave function gives "the amplitude for that three-geometry to arise from a zero three-geometry i.e. a single point. In other words the ground state is the amplitude for the universe to appear from nothing"

Harte-Hawking wavefunction in the minisuperspace approximation with $\Lambda > 0$

$$\Psi = \int \delta g_{\mu\nu}\; e^{iS_{EH} + iS_B} \qquad ds^2 = -\frac{N^2}{q(t)} dt^2 + q(t)\, d\Omega_3^2$$

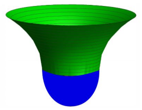


$$\ddot{q} = 0$$

$$3p^2 + 3 - \Lambda q = 0 \qquad q_0 = 0 \qquad p_0 = \pm i$$

Dirichlet bcs $\quad q_0 = 0 \quad q_1$

$$S_B = \int_{out} \sqrt{h} K - \int_{in} \sqrt{h} K$$

$\Psi \approx$  

Mixed bcs $\quad p_0 = +i \quad q_1$

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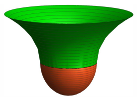
$\Psi \approx$  

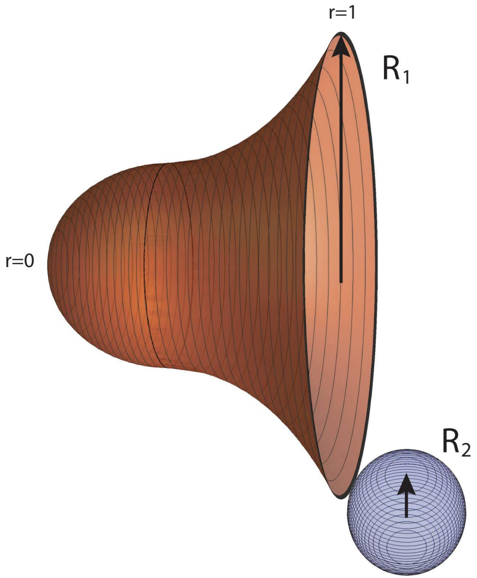
[Halliwell Louko '89, Feldbrugge Lehners Turok '17, **DT** Lehners '18 , **DT** Lehners Sberna '19]

Hawking-Page phase transition from path integrals in the minisuperspace approximation with $\Lambda = -\frac{3}{l^2} < 0$

Partition function as a sum of metrics with fixed Euclidean boundary $S^1 \times S^2$

$$Z = \int \delta g_{\mu\nu}\, e^{iS_{EH}+iS_B}$$

Kantowski-Sachs ansatz $ds^2 = -\frac{b(r)}{c(r)} N^2 dr^2 + \frac{c(r)}{b(r)} dt^2 + b(r)^2 d\Omega_2^2$



Schwarzschild BH in EAdS for:

$$b(r) = (R_2 - r_+)r + r_+ \qquad c(r) = \frac{b(r)^3}{l^2} + b(r) - 2M$$

$$N = \pm i(R_2 - r_+) \qquad 2Ml^2 = r_+(l^2 + r_+^2)$$

$$\beta = \frac{4\pi l^2 r_+}{l^2+3r_+^2}$$

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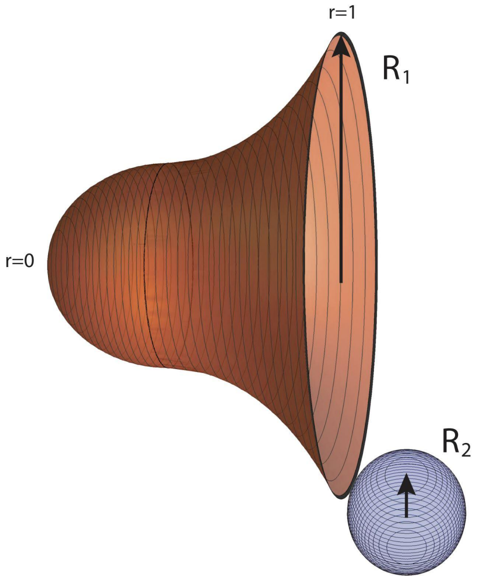


<u>Boundary conditions:</u>

Dirichlet at the "outer boundary":

We fix the radius of the circle and the 2-sphere $q(1) = R_2$ $\sqrt{\frac{c(1)}{b(1)}}\beta = R_1$

Neumann at the "inner boundary": no boundary term

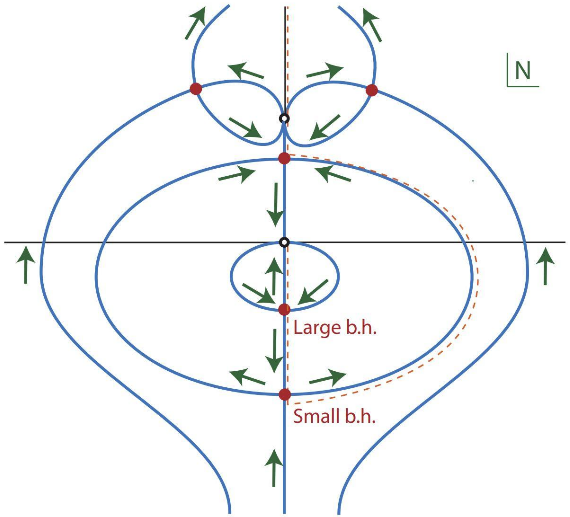
The geometry starts at the horizon $c(0) = 0$ with no conical deficit $\frac{\dot{c}}{Nb}\Big|_{r=0} = \frac{4\pi i}{\beta}$ (= at the BH sps $\beta = \frac{4\pi l^2 r_+}{l^2+3r_+^2}$)

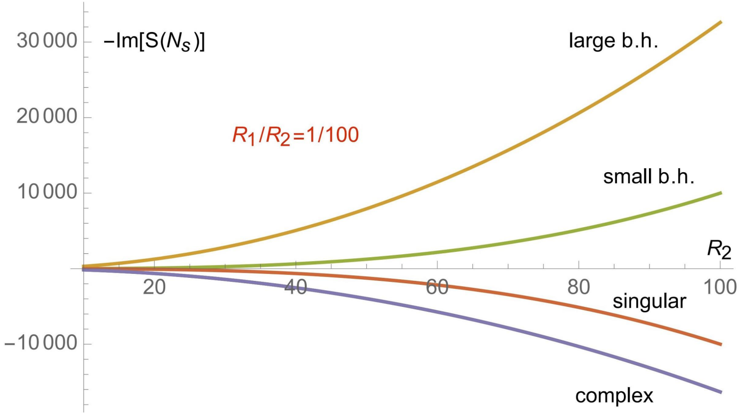
⟶ Neumann condition as a regularity condition as in cosmology

Evaluation of the path integral

Gaussian integral over c and b , steepest descent analysis in the complex N plane (numerically)

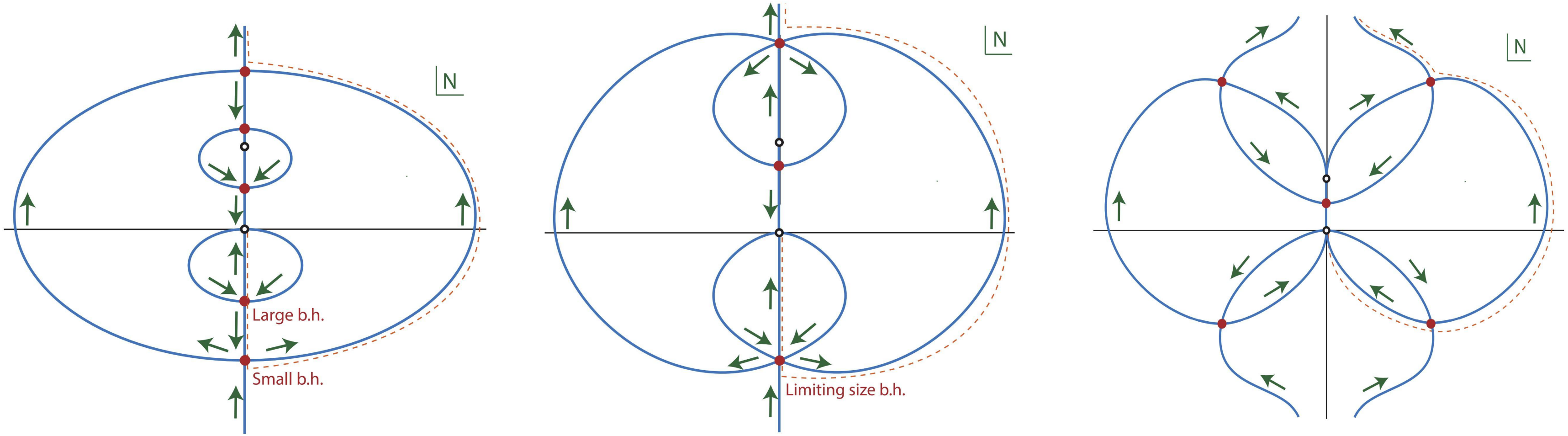
For high temperature 5 saddle points:
1 large and 1 small EAdS black holes, 2 complex geometries, 1 irregular geometry





Only the blak holes saddle points survive when the boundary is pushed to infinity

Lowering the temperature



$$\frac{R_1}{R_2} = \frac{2\pi}{\sqrt{3}}\sqrt{1 + \frac{l^2}{R_2} - \frac{4l^2}{3\sqrt{3}R_2^3}}$$

$$\beta = \frac{2\pi l}{\sqrt{3}}$$

There is a minimum temperature for which Euclidean black hole solutions exist.
Below this temperature the corresponding saddle points become complex.

Thermodynamics from saddles

$$\Delta S = S_{BH} - S_{AdS} = -\frac{iR_1}{4Gl}\left[\frac{\sqrt{R_2}(4R_2^3+4l^2R_2-3l^2r_+-r_+^3)}{\sqrt{R_2^3+l^2R_2-l^2r_+-r_+^3}} - 4R_2\sqrt{R_2^2+l^2}\right] = -\frac{i\pi}{G}r_+^2\frac{r_+^2-l^2}{3r_+^2+l^2} + O(R_2^{-1})$$

AdS solution dominates for $r_+ < l$ $\quad (R_1 \approx \pi R_2\ )$ ⟶ complex saddles never dominate

$$\ln Z = \frac{i\Delta S}{\hbar} = \frac{R_2}{Tl_p^2}\left(\sqrt{1+\frac{R_2^2}{l^2}-\frac{2M}{R_2}} - \sqrt{1+\frac{R_2^2}{l^2}}\right) + \frac{\pi r_+^2}{l_p^2}$$

$$\langle E\rangle = k_B T^2\frac{\partial \ln Z}{\partial T} = \frac{k_B}{l_p^2}\frac{lM}{R_2} - \frac{k_B}{l_p^2}\frac{Ml^3}{2R_2^3} + O(R_2^{-4})$$

$$\mathcal{S} = k_B \ln Z + \frac{\langle E\rangle}{T} = \frac{k_B}{l_p^2}\pi r_+^2 = \frac{k_B}{l_p^2}\frac{Area}{4}$$

$$-k_B T\ln Z = \langle E\rangle - T\mathcal{S}$$

$$\mathcal{M} = \sqrt{1+\frac{R_2^2}{l^2}-\frac{2M}{R_2}}\langle E\rangle = M + \frac{l^2}{2R_2^2}M - \frac{l^2}{R_2^3}M^2 + O(R_2^{-4})$$

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It all works well because we added
no GHY boundary term in the interior i.e.
we imposed a **Neumann** boundary condition

Add we used a Dirichlet condition we would have obtained an additional boundary term contribution to the BH action of magnitude
$\frac{k_B}{l^2}\pi r_+^2$

$$-k_B\ln Z \approx \frac{\langle E\rangle}{T} - \mathcal{S} + \mathcal{S} = \frac{\langle E\rangle}{T}$$

S^3 partition function, $\Lambda < 0$

$$Z = \int \delta g_{\mu\nu}\, e^{i(S_{E-H} + \int_{out} \sqrt{h} K + S_{ct})} \qquad ds^2 = -\frac{N^2}{q(r)} dr^2 + q(r) d\Omega_3^2$$

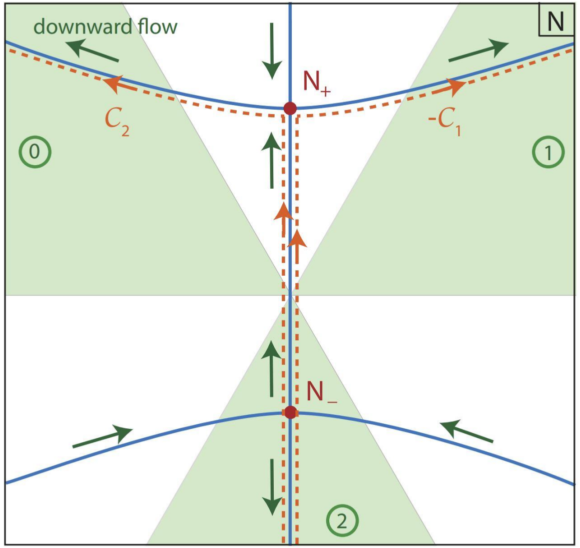
Mixed boundary conditions: $q(1) = R_3 \qquad p(0) = \alpha$

$$Z(R_3) = e^{i\frac{V_3}{8\pi G\hbar}\alpha(3+\alpha^2)l^2}\, Bi\left[\left(\frac{3V_3}{8\pi G\hbar l}\right)^{2/3}(R_3^2 + l^2)\right] e^{\frac{i}{\hbar}S_{ct}}$$

for $R_3 \to \infty \qquad Z(R_3) = e^{i\frac{V_3}{8\pi G\hbar}\alpha(3+\alpha^2)l^2}$

Three-sphere partition function of ABJM theory $Z = Ai\left[\left(\frac{3V_3 l^2}{8\pi G\hbar}\right)^{2/3}\right] \approx e^{-\frac{2V_3 l^2}{8\pi G\hbar}}$

Then we must impose $\alpha = i$!



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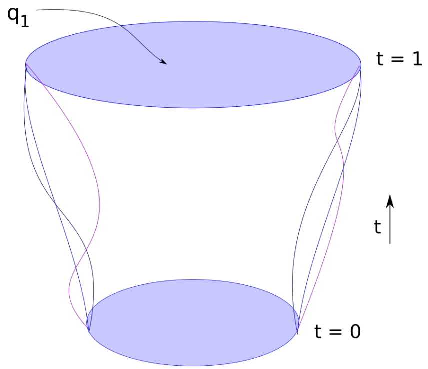

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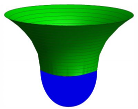
$\Psi \approx$  

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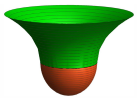
$\Psi \approx$  

[Halliwell Louko ‘89, Feldbrugge Lehners Turok ‘17, **DT** Lehners ‘18 , **DT** Lehners Sberna ‘19]

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$$Z = \int \delta g_{\mu\nu}\, e^{i(S_{E-H} + \int_{out} \sqrt{h}K - \int_{in} \sqrt{h}K + S_{ct})}$$

$$ds^2 = -\frac{N^2}{q(r)}dr^2 + q(r)d\Omega_3^2$$

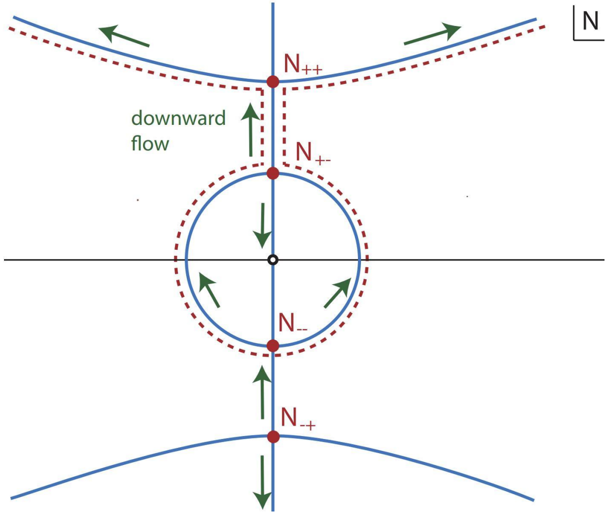
Dirichlet boundary conditions: $q(1) = R_3 \qquad q(0) = 0$

$$Z \propto Ai\left[\left(\frac{3V_3 l^2}{8\pi G\hbar}\right)^{2/3}\right] Bi\left[\left(\frac{3V_3 l^2}{8\pi G\hbar}\right)^{2/3}(R_3^2 l^2)\right] e^{iS_{ct}} + $$
$$+ Bi\left[\left(\frac{3V_3 l^2}{8\pi G\hbar}\right)^{2/3}\right] Ai\left[\left(\frac{3V_3 l^2}{8\pi G\hbar}\right)^{2/3}(R_3^2 + l^2)\right] e^{iS_{ct}}$$

for $R_3 \to \infty$

$$Z_D = Ai\left[\left(\frac{3V_3 l^2}{8\pi G\hbar}\right)^{2/3}\right]$$

$$Z_N = e^{-\frac{2V_3 l^2}{8\pi G\hbar}}$$



[Caputa Hirano '18]

Summary

- Saddle point approximation can be subtle. Which path integral is approximated by a given saddle point? Does this fit with the physical situation?
- Minisuperspace is useful for exact results. It’s validity must be verified!
- Additional saddle points for AdS cut at a finite radius: what do they mean in qft?
- Nor Euclidean nor Lorentzian AdS path integral
- Mixed boundary conditions. Neumann in the interior as a regularity condition
- Black holes thermodynamics supports the no-boundary proposal