Decoding the Path Integral: Resurgence and Non-perturbative Physics

Gerald Dunne
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Theory Seminar, King's College London, April 6, 2022

Physical Motivation

the Feynman path integral is both conceptually and computationally powerful … but

… some important physics computations are still challenging

- finite density: e.g. the "sign problem"
- non-equilibrium physics at strong-coupling
- real time evolution
- quantum systems in extreme background fields (gauge and gravitational)

standard computational methods from path integrals

- perturbation theory
- non-perturbative numerical methods: Monte Carlo
- non-perturbative semi-classical methods: "instantons"
- asymptotics

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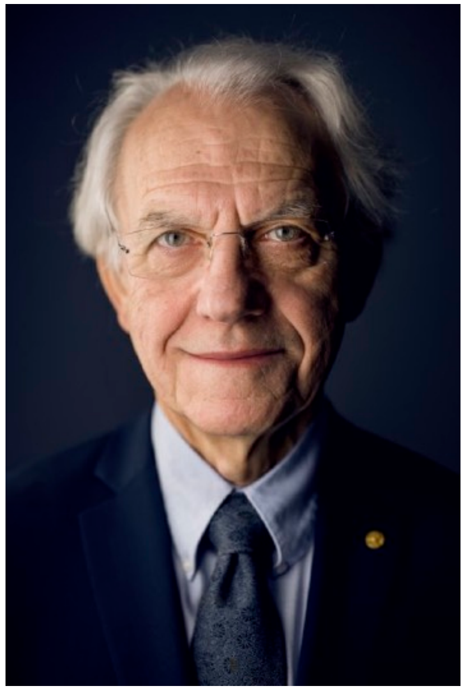
"resurgence": seeks to unify these approaches

technical problem: how to actually compute a quantum path integral?

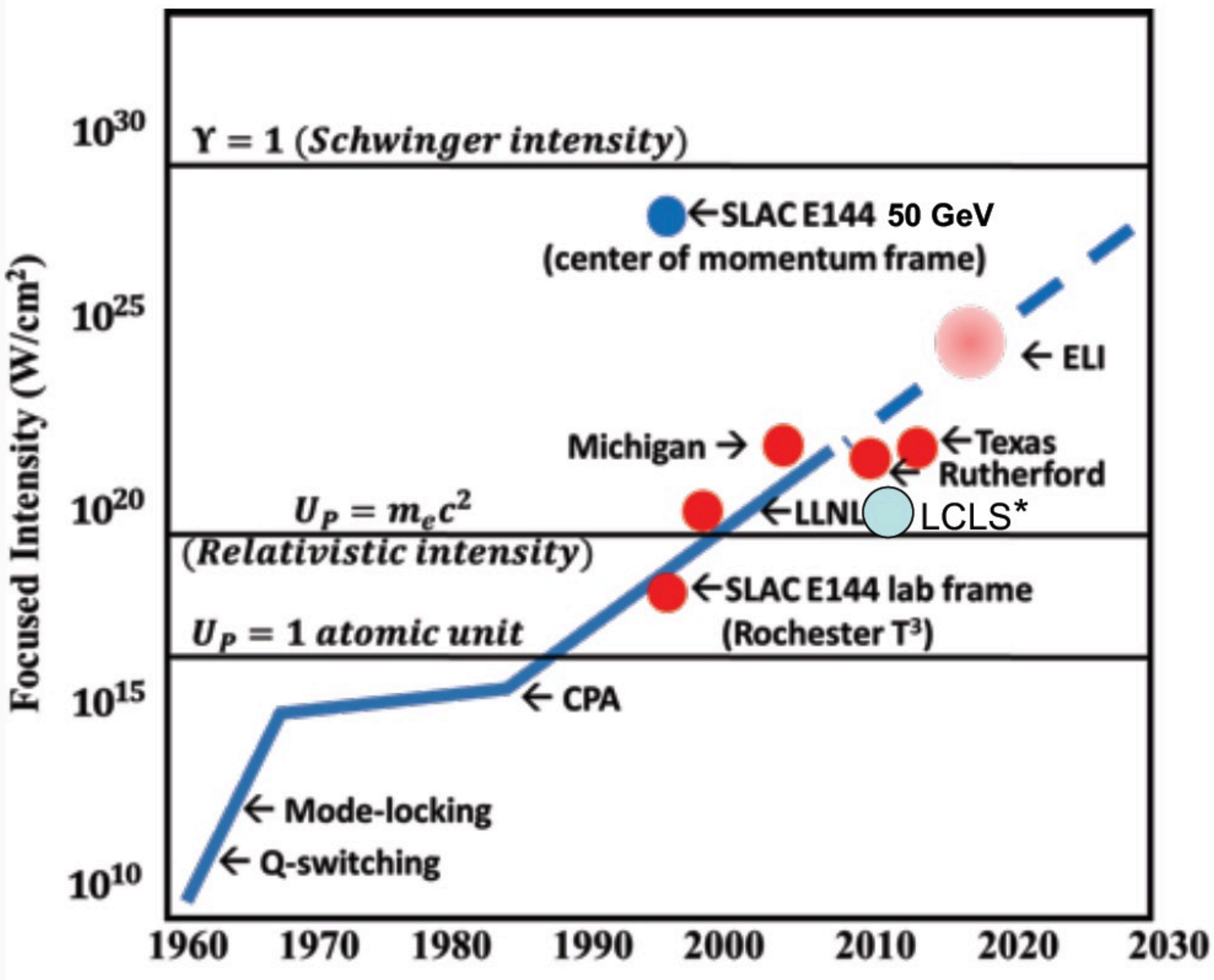
[QFT at Extreme Intensities](#)

Strickland
& Mourou
Nobel Prize 2018



- Current experimental proposals: laser-laser; laser-lepton; lepton-lepton; highly-charged ions; …
- Important theoretical puzzles remain
- Astro-physical applications
- Semiclassical computations
- Non-equilibrium physics
- Ultra-fast dynamics

- FACET II (SLAC): [Snowmass Letter of Intent](#)



Reviews: Di Piazza et al, 2012;
US National Academies, 2018

Physical motivation: Non-perturbative intense-field QFT

Understanding the Fully Non-Perturbative Strong-Field Regime of QED.

(Letter of Intent to Snowmass Theory Frontier)

Philip H. Bucksbaum, Gerald V. Dunne[*], Frederico Fiuza, Sebastian Meuren[†], Michael E. Peskin, David A. Reis,

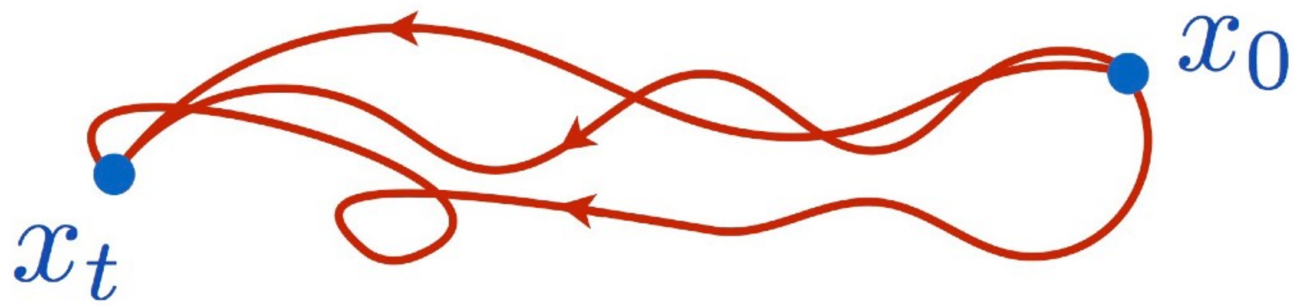
Greger Torgrimsson, Glen White, and Vitaly Yakimenko

June 25, 2020

Abstract: Although perturbative QED with small background fields is well-understood and well-tested, there is much less understanding of QED in the regime of strong fields. At the Schwinger critical field, $E_c = m^2c^3/eh = 10^{18}$ V/m, the vacuum becomes unstable to pair production. For stronger fields, such that $\alpha(E/E_c)^{2/3} > 1$, QED perturbation theory breaks down. These regimes are relevant to environments found in high-energy astrophysics and to physics in the collisions of high-energy electron and heavy ion beams. We plan to investigate problems in beam simulation and basic QED theory related to QED at strong fields, to analyze experiments that can test QED in this regime, and to explore applications to astrophysics and particle physics.

The Feynman Path Integral

$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



$$\text{QM:} \quad \int \mathcal{D}x(t)\ \exp\left[\frac{i}{\hbar} S[x(t)]\right]$$

$$\text{QFT:} \quad \int \mathcal{D}A(x^\mu)\ \exp\left[\frac{i}{g^2} S\left[A(x^\mu)\right]\right]$$

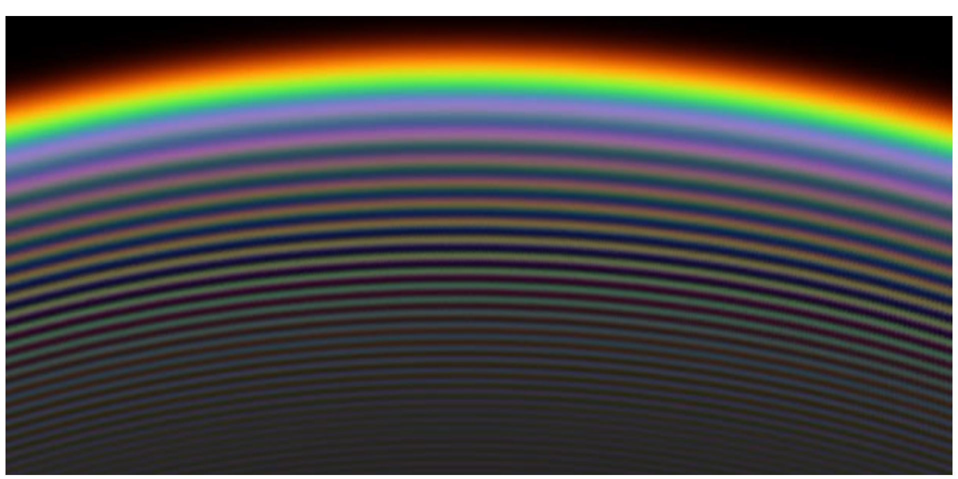
- stationary phase approximation: classical physics
- bridge from classical to quantum field theory
- quantum perturbation theory: fluctuations about trivial saddle point
- other saddle points: non-perturbative physics
- resurgence: saddle points are related by analytic continuation, so perturbative and non-perturbative physics are *unified*

Resurgence in Classical Optics: the original "sign problem"

Airy, Stokes and *spurious/supernumerary* rainbows

Airy 1838: "On the intensity of light in the neighbourhood of a caustic"

Stokes 1850: "On the numerical calculation of a class of definite integrals and infinite series"



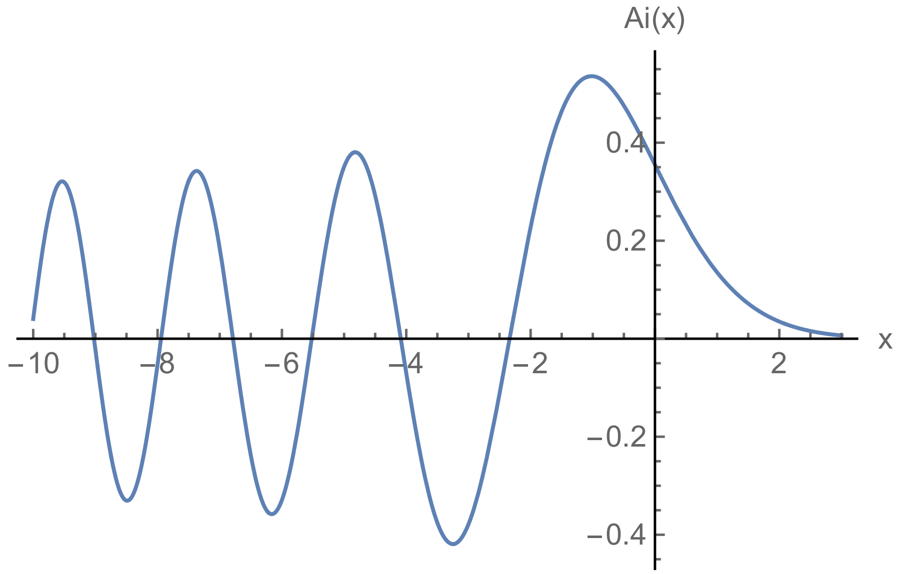
W. Miller 1841: "On Spurious Rainbows"

"Stokes, by mathematical supersubtlety transformed Airy's integral into a form by which the light at any point of any of those thirty bands could be calculated with but little effort …"
Lord Kelvin (Stokes obituary, 1903)

The Stokes Phenomenon

Stokes 1857: "On the discontinuity of arbitrary constants which appear in divergent developments"

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, e^{i\left(\frac{1}{3}t^3 + x\,t\right)}$$



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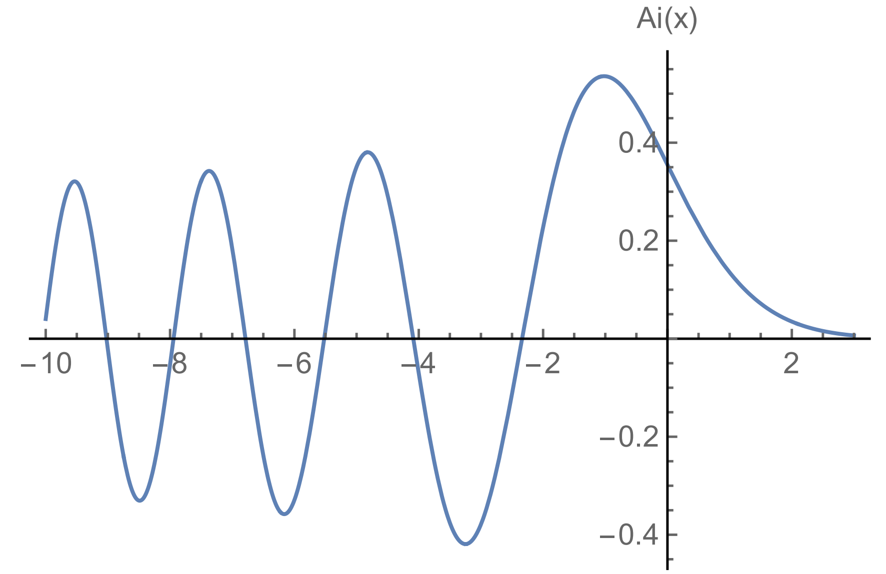
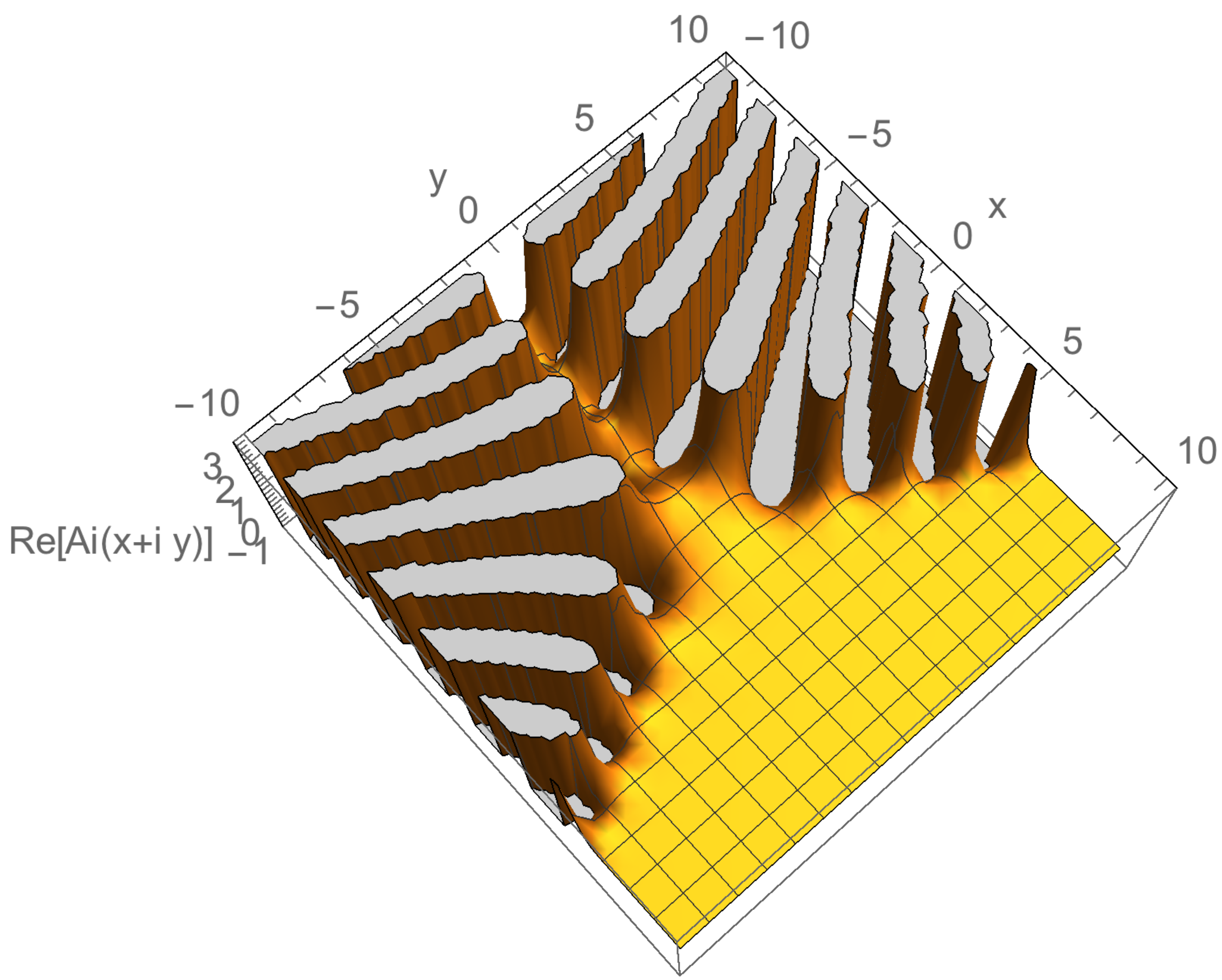
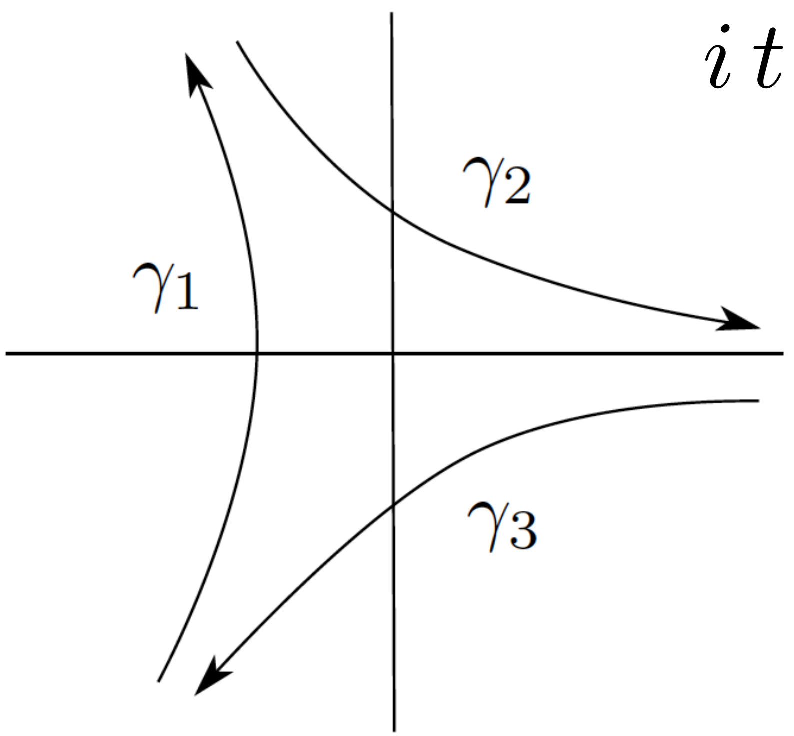


Stokes transitions occur in the complex *x* plane





non-perturbative connection formulae connect sectors

$$\mathrm{Bi}(x) = 2\, e^{\pm\frac{\pi i}{6}}\,\mathrm{Ai}\left(e^{\mp\frac{2\pi i}{3}}\, x\right) \mp i\,\mathrm{Ai}(x)$$

Analytic Continuation of Path Integrals

since we need complex analysis and contour deformation to make sense of oscillatory exponential integrals, it is natural to explore similar methods for (infinite dimensional) path integrals

$$\int \mathcal{D}x(t)\, \exp\left[\frac{i}{\hbar}\, S[x(t)]\right] \longleftrightarrow \int \mathcal{D}x(t)\, \exp\left[-\frac{1}{\hbar}\, S[x(t)]\right]$$

goal: a satisfactory formulation of the functional integral should be able to describe Stokes transitions

idea: seek a computationally viable constructive definition of the path integral using ideas from resurgent trans-series

Trans-Series

- an interesting observation by Hardy:

"The only scales of infinity that are of any practical importance in analysis are those which may be constructed by means of the logarithmic and exponential functions."

G. H. Hardy, *Orders of Infinity,* 1910

- deep result: "this is all we need" (J. Ecalle, 1980s)

- trans-series is generated by iterations of "trans-monomials"

$$\hbar,\ e^{-1/\hbar},\ \ln\hbar$$

- conjecture: trans-series practically sufficient "for all natural problems"

- observation: this structure matches the asymptotics of all physics computations; e.g., perturbative and non-perturbative QFT computations

Resurgent Trans-Series

Ecalle 1980s
Dingle 1960s
Stokes 1850s

resurgence: new-ish idea in mathematics

perturbative series $\longrightarrow$ "trans-series"

physics applications: "semiclassical trans-series"

$$f(\hbar) = \sum_p c_{[p]}\, \hbar^p \quad \longrightarrow \quad f(\hbar) = \sum_k \sum_p \sum_l c_{[kpl]}\, e^{-\frac{k}{\hbar}}\, \hbar^p\, (\ln \hbar)^l$$

- trans-series is well-defined under analytic continuation
- well understood for differential/difference/integral equations & exponential integrals: "all natural problems"
- expansions about different saddles are related
- exponentially improved asymptotics

physics: necessarily unifies perturbative and non-perturbative physics

Resurgent Functions

"resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities"

J. Ecalle, 1980

physical implication: fluctuations about different sectors are related

conjecture: this structure is general

Resurgence in Exponential Integrals

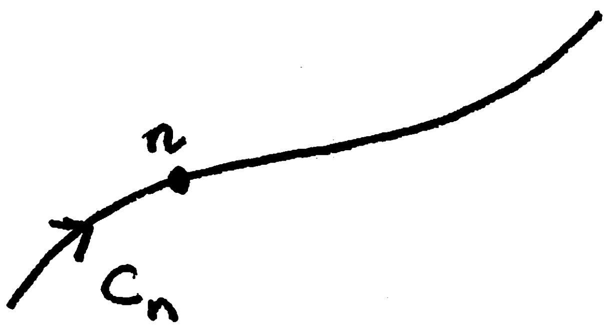
Dingle 1960s;
Berry & Howls 1991:
"Hyperasymptotics for Integrals with Saddles"

steepest descent integral through saddle point "n":

$$I^{(n)}(\hbar) = \int_{C_n} dx\, e^{\frac{i}{\hbar} f(x)} = \frac{1}{\sqrt{1/\hbar}}\, e^{\frac{i}{\hbar} f_n}\, T^{(n)}(\hbar)$$

all fluctuations beyond the Gaussian approximation:

$$T^{(n)}(\hbar) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, \hbar^r$$



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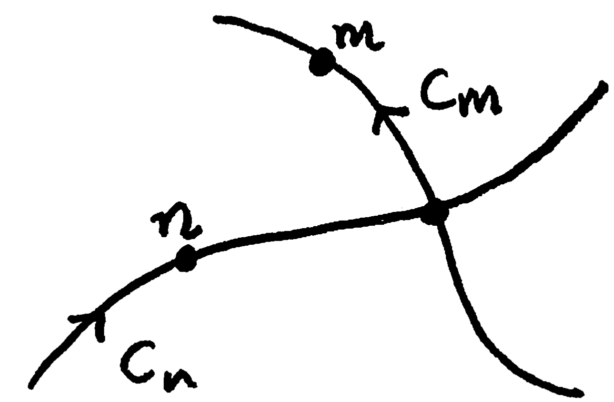

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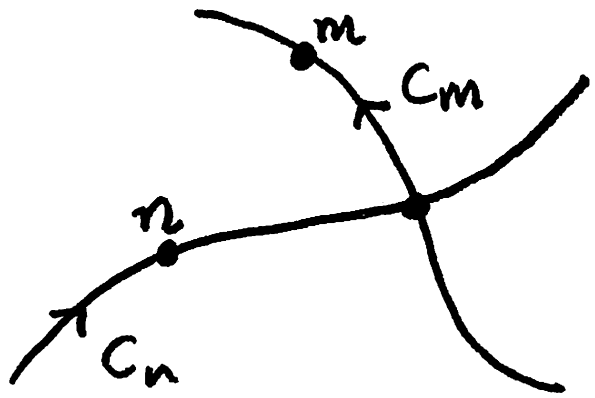

straightforward complex analysis implies:

universal large orders of fluctuation coefficients: $\left(F_{nm} \equiv f_m - f_n\right)$

$$T_r^{(n)} \sim \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(\pm 1)}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)}\, T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)}\, T_2^{(m)} + \ldots \right]$$

fluctuations about different saddles are quantitatively related

Resurgence in Exponential Integrals

canonical example: Airy function integral has 2 saddle points: +/-

$$T_r^{\pm} = (\pm 1)^r \frac{\Gamma\left(r+\frac{1}{6}\right)\Gamma\left(r+\frac{5}{6}\right)}{(2\pi)\left(\frac{4}{3}\right)^r r!} = \left\{1, \pm\frac{5}{48}, \frac{385}{4608}, \pm\frac{85085}{663552}, \dots\right\}$$

large orders of fluctuation coefficients:

$$T_r^{+} \sim \frac{(r-1)!}{(2\pi)\left(\frac{4}{3}\right)^r}\left(1 - \left(\frac{4}{3}\right)\frac{5}{48}\frac{1}{(r-1)} + \left(\frac{4}{3}\right)^2\frac{385}{4608}\frac{1}{(r-1)(r-2)} - \dots\right)$$

generic "large-order/low-order" resurgence relation

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generic "large-order/low-order" resurgence relation

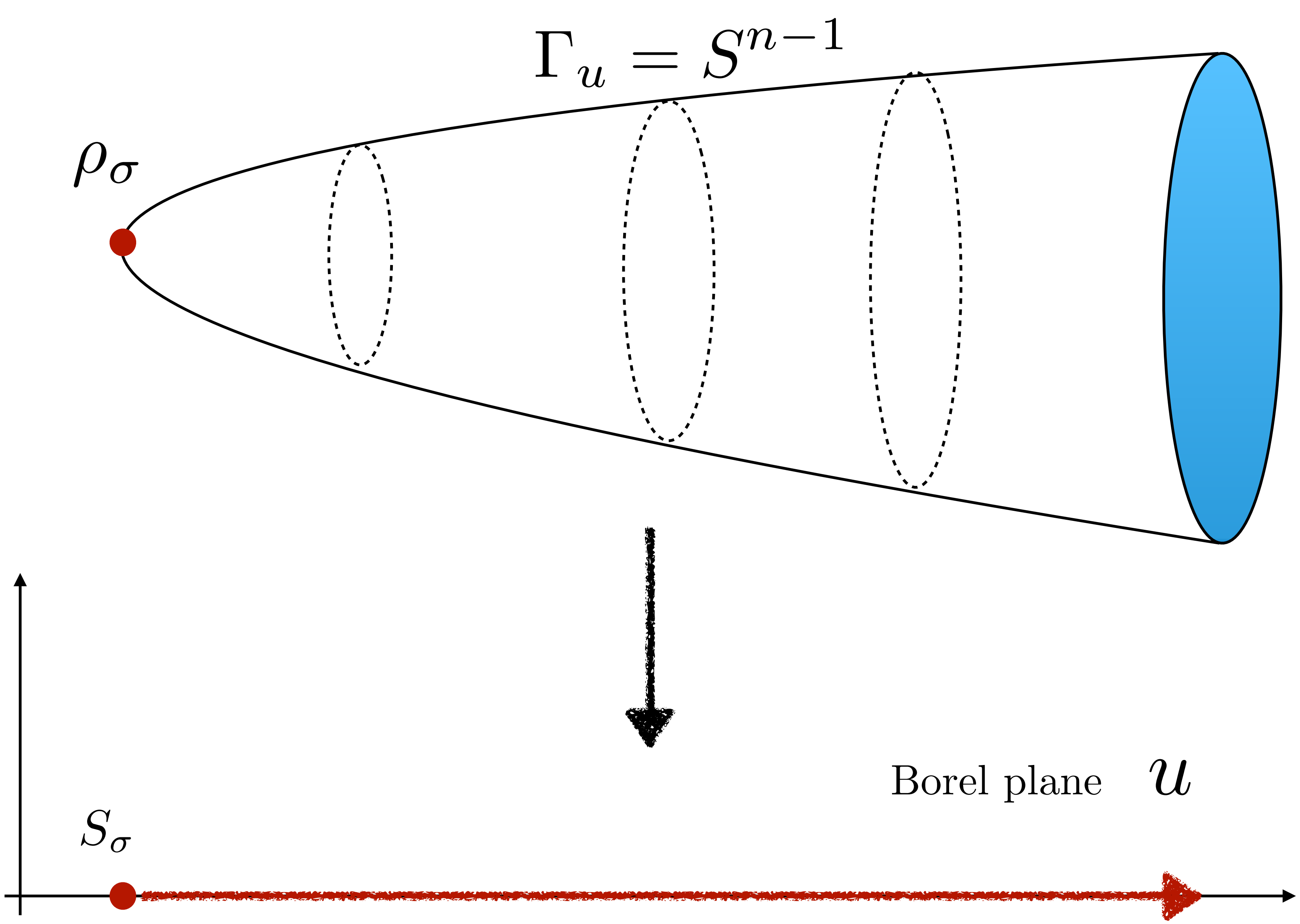
remarkable fact: this resurgent large-order/low-order behavior has been found in matrix models, QM, QFT, string theory, …

the natural way to explain this is via analytic continuation of functional integrals

Lefschetz Thimbles

even the generalization to more than 1 complex dimension is interesting

Pham 1967;
Howls 1997,
Kontsevich 2020, …



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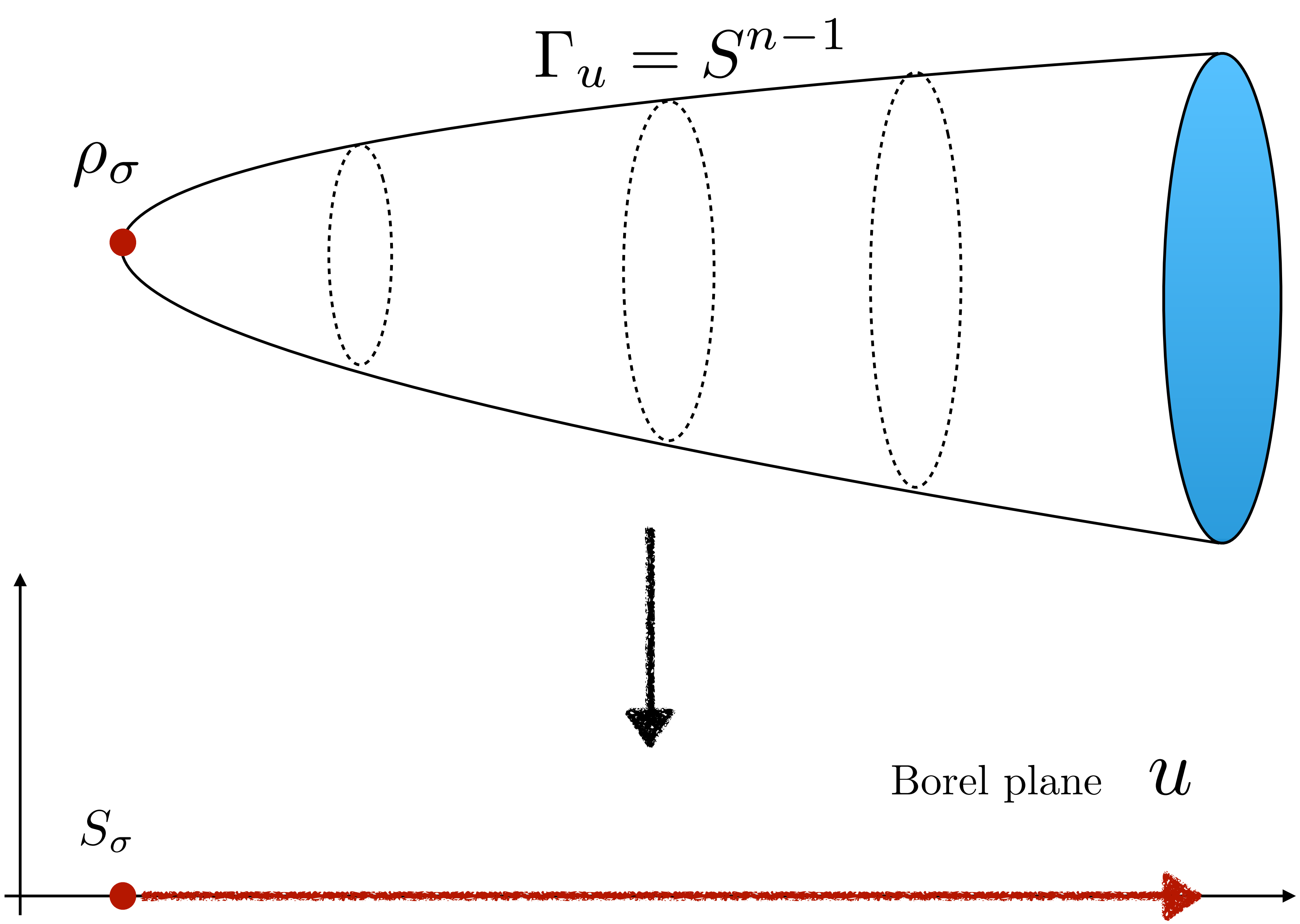
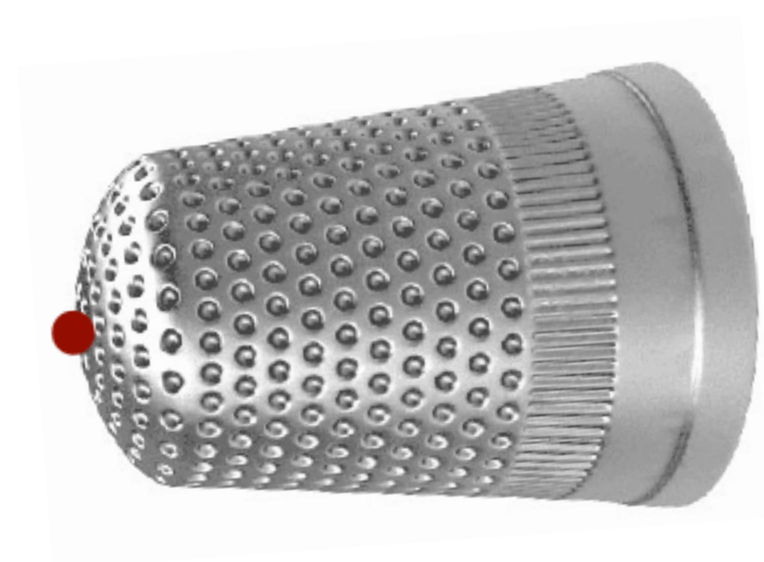




Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \overset{?}{=} \sum_{\text{thimble}} \mathcal{N}_{\text{th}} \, e^{i\,\phi_{\text{th}}} \int_{\text{th}} \mathcal{D}A \; \times (\mathcal{J}_{\text{th}}) \times \exp\left(\mathcal{R}e\left[\frac{i}{\hbar} S[A]\right]\right)$$

Lefschetz thimble = "functional steepest descents contour"

- in principle, on a thimble, the path integral becomes well-defined and computable

- complexified gradient flow:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

CRISTOFORETTI *et al.* (2013)

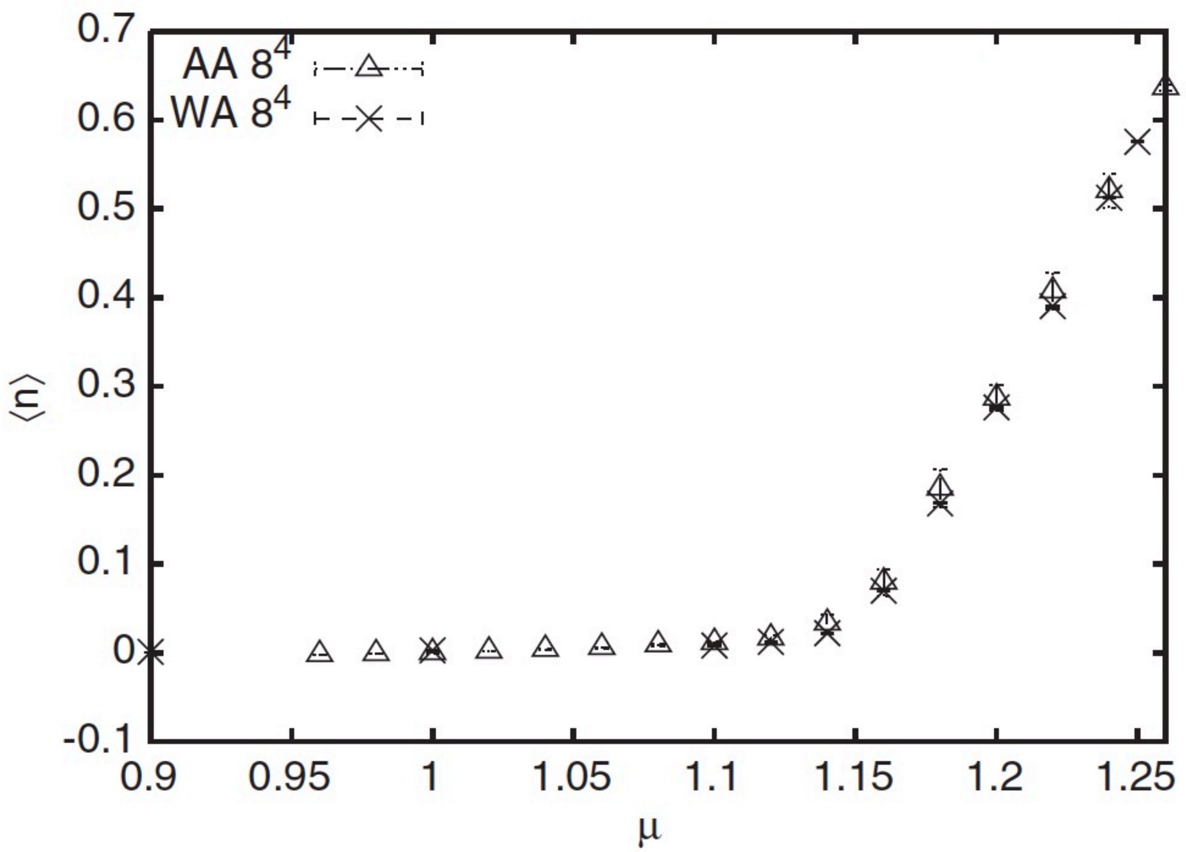
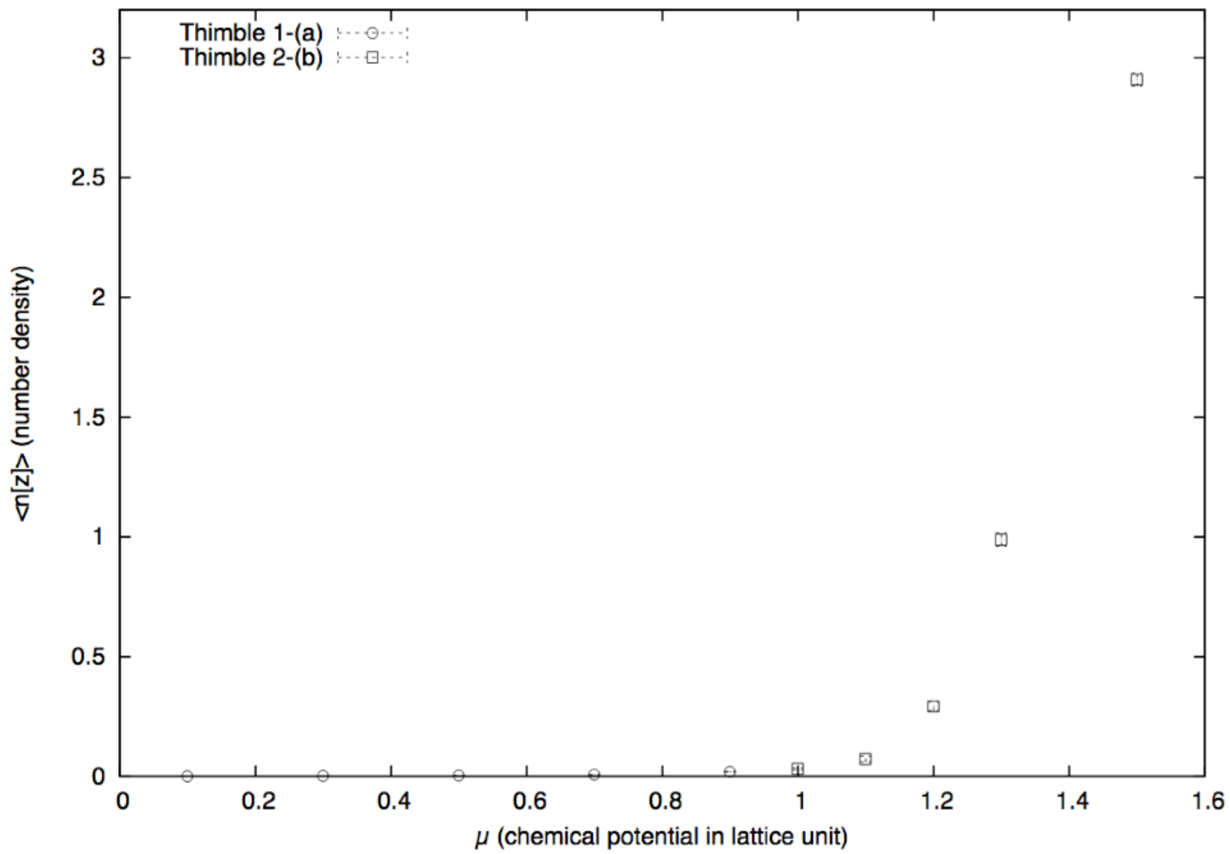


FIG. 3. Comparison of the average density $\langle n \rangle$ obtained with the worm algorithm (WA) [22] with the Aurora algorithm (AA)

Fujii et al (2013)



- 4d relativistic Bose gas: complex scalar field theory
- Monte Carlo on a thimble softens the sign problem
- results comparable to "worm algorithm"

Phase Transitions in QFT: 2d Thirring Model

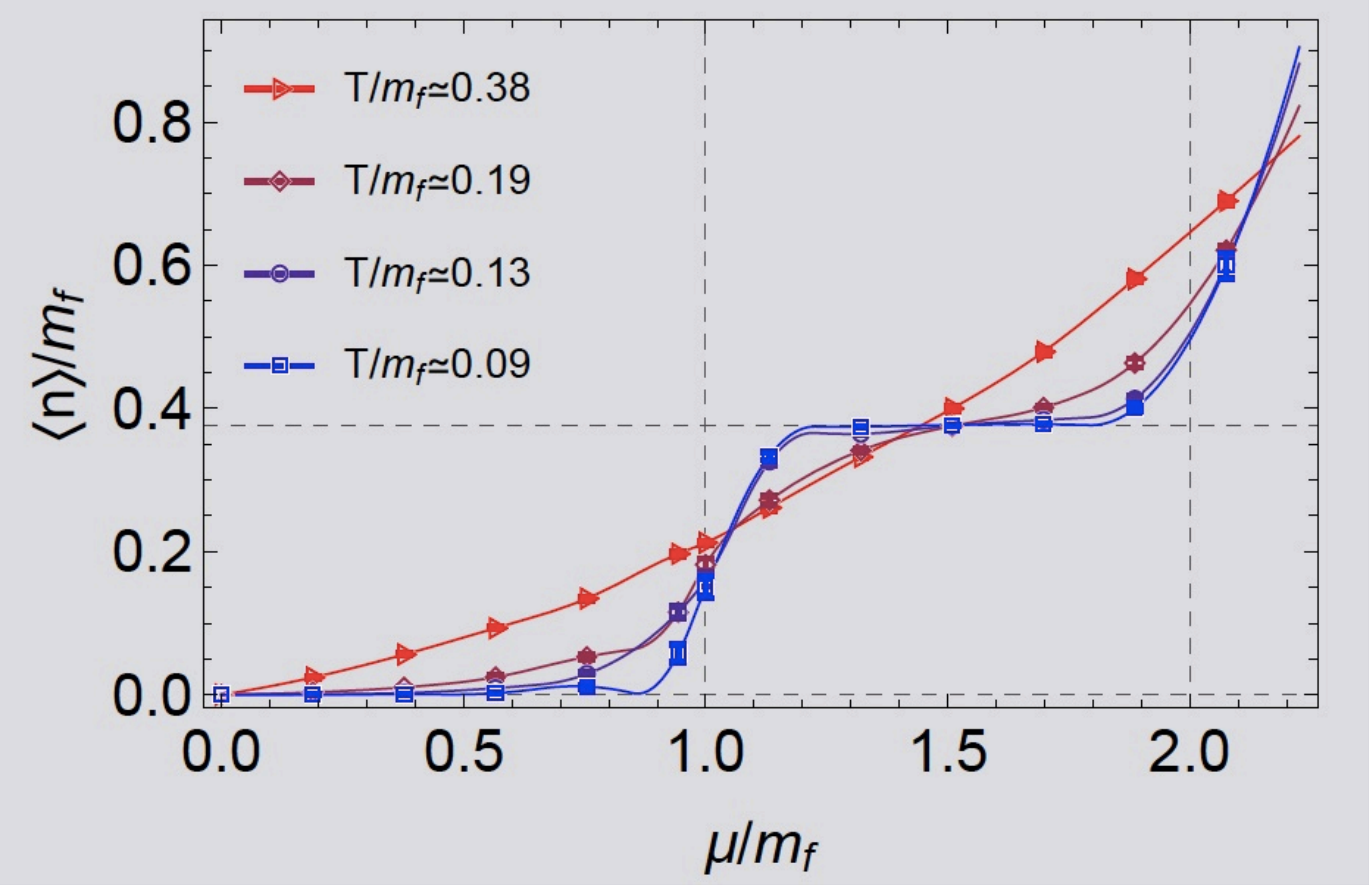
(Alexandru et al, 2016)

$$\mathcal{L} = \bar{\psi}^a \left(\gamma_\nu \partial_\nu + m + \mu\,\gamma_0\right) \psi^a + \frac{g^2}{2N_f} \left(\bar{\psi}^a \gamma_\nu \psi^a\right) \left(\bar{\psi}^b \gamma_\nu \psi^b\right)$$

- chain of interacting fermions: asymptotically free
- sign problem at nonzero density
- generalized thimble method: balance flow cost with sign problem cost

Monte Carlo taming of the sign problem and demonstration of the "Silver Blaze" phenomenon

Rev. Mod. Phys. 2022



Resurgence in QM or QFT Path Integrals?

- perturbation theory works, but it is generically divergent, and it is only part of the story

- resurgence: perturbation theory encodes non-perturbative information

path integral → perturbation theory → Borel → trans-series

path integral → saddle expansion → asymptotics → trans-series

path integral → Lefschetz thimbles/Monte Carlo → trans-series

main conjecture: these should all be the same thing, and resurgence should connect them, as well as connecting different saddles

PHYSICAL REVIEW VOLUME 85, NUMBER 4 FEBRUARY 15, 1952

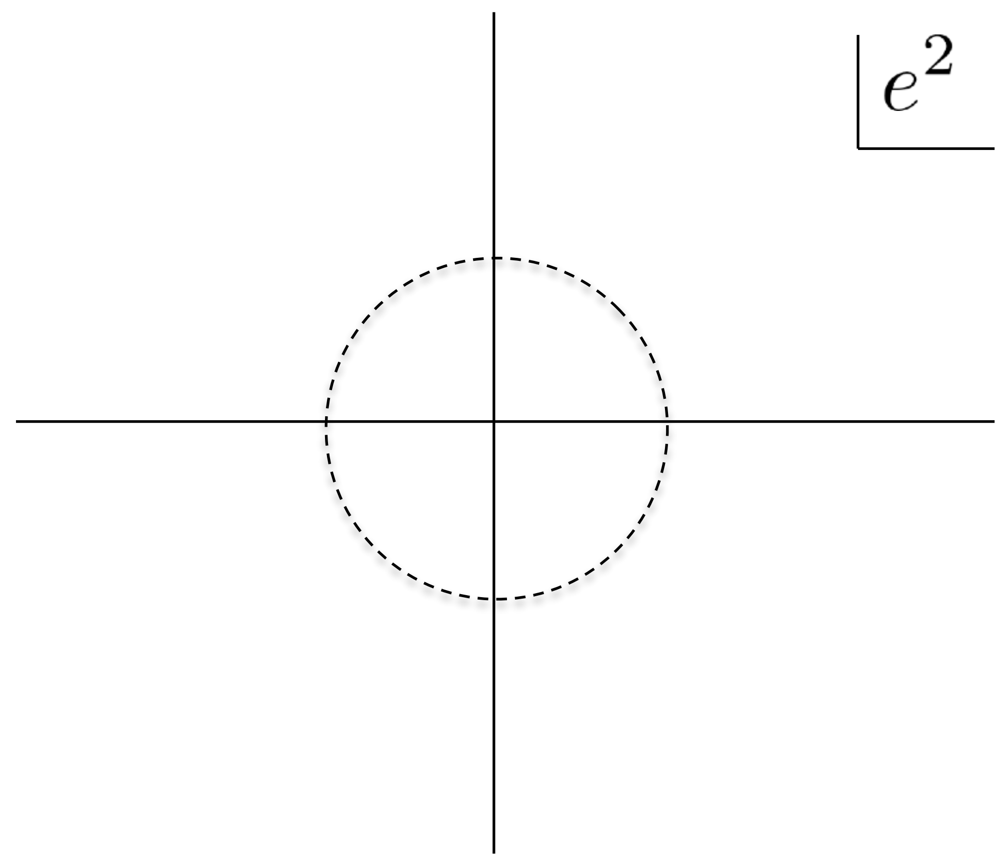
Divergence of Perturbation Theory in Quantum Electrodynamics

F. J. Dyson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

(Received November 5, 1951)

An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge. The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built.

$$F = a_0 + a_1 e^2 + a_2 e^4 + a_3 e^6 + \ldots$$



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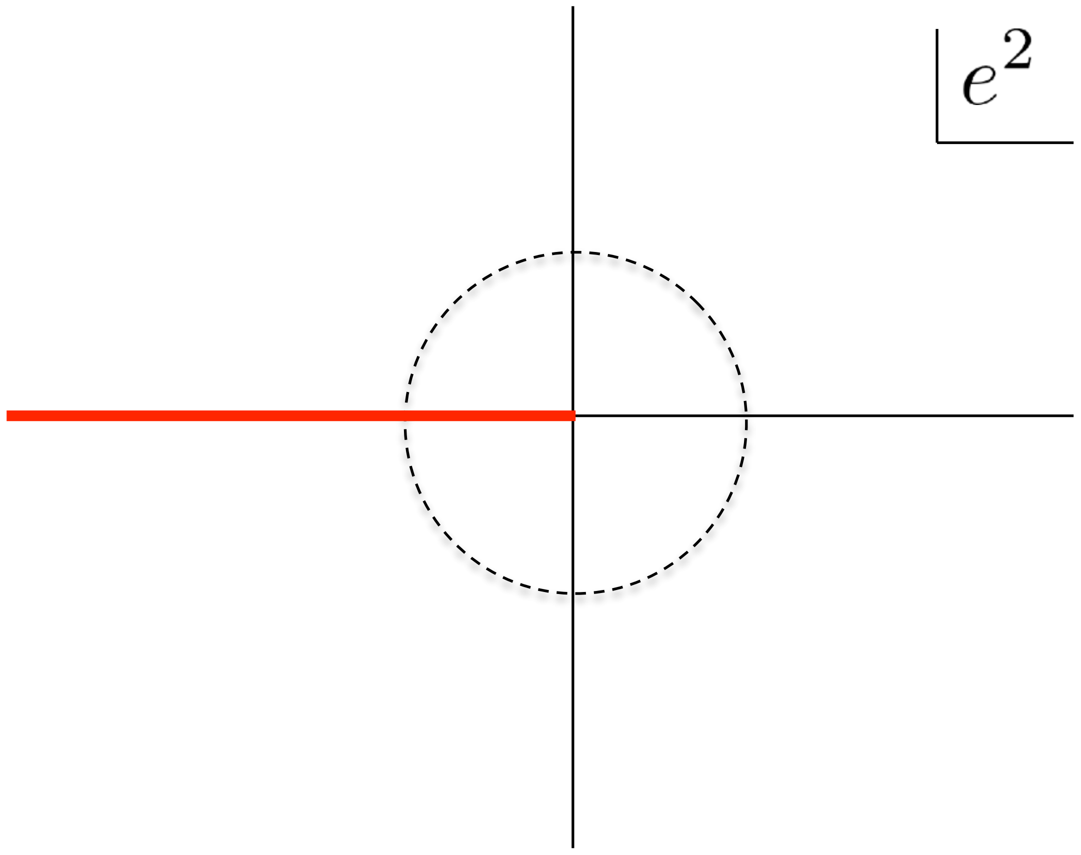

$e^2 < 0$

unstable

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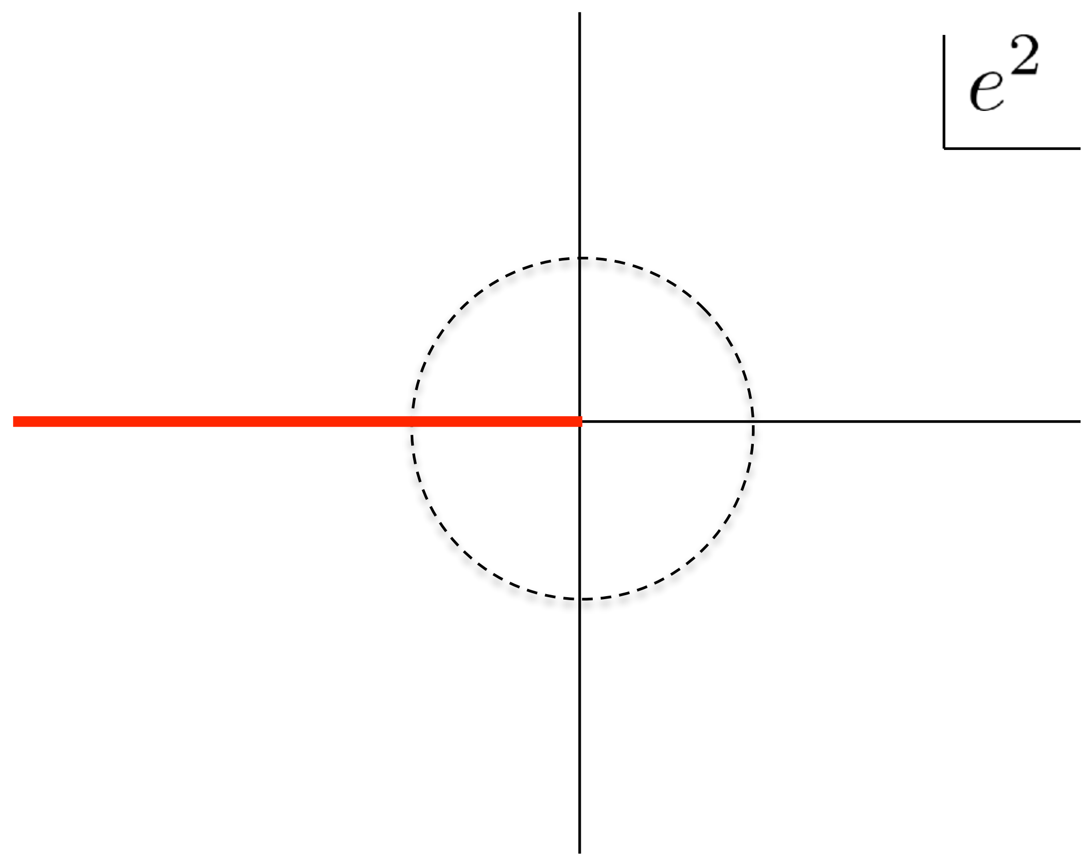


$e^2 < 0$

unstable

"The argument here presented is lacking in mathematical rigor and in physical precision. It is intended only to be suggestive, to serve as a basis for further discussions. … Also I am glad to have this opportunity to withdraw the erroneous argument previously put forward to support the claim that the power series should converge."

Borel Summation: Physical Regularization of Divergent Series

Borel transform of a divergent series with $c_n \sim n!$

$$f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n \qquad \rightarrow \qquad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \, t^n$$

Borel sum of the divergent series:

$$\mathcal{S}[f](g) = \frac{1}{g} \int_0^{\infty} dt \, e^{-t/g} \, \mathcal{B}[f](t)$$

- the singularities of **B**[f](t) provide a physical encoding of the global asymptotic behavior of f(g)
- singularities of Borel transform ⟷ non-perturbative physics
- singularities on positive Borel t axis: exponentially small imaginary part

The Physics of Borel Summation

- computing perturbative coefficients is difficult
- generic Bender-Wu-Lipatov growth rate of perturbative coefficients:

$$c_n \sim \mathcal{S}\, \frac{\Gamma(a\, n + b)}{A^n} \qquad , \qquad n \to \infty$$

- A: *location* of leading Borel singularity: "leading instanton action"
- b : *fluctuation exponent* about leading instanton
- a : determines the appropriate *expansion variable ("Ecalle variable")*
- S : Stokes constant = *strength* of leading instanton effect

- in fact, resurgence implies that there is much more information encoded in the perturbative coefficients

QM Perturbation Theory: Zeeman & Stark Effects

Zeeman : divergent, *alternating*, asymptotic series

$$a_n \sim (-1)^n (2n)!$$

Borel singularities on the negative Borel axis.

physics: Magnetic field causes (real) energy level shifts

Stark : divergent, *non-alternating*, asymptotic series

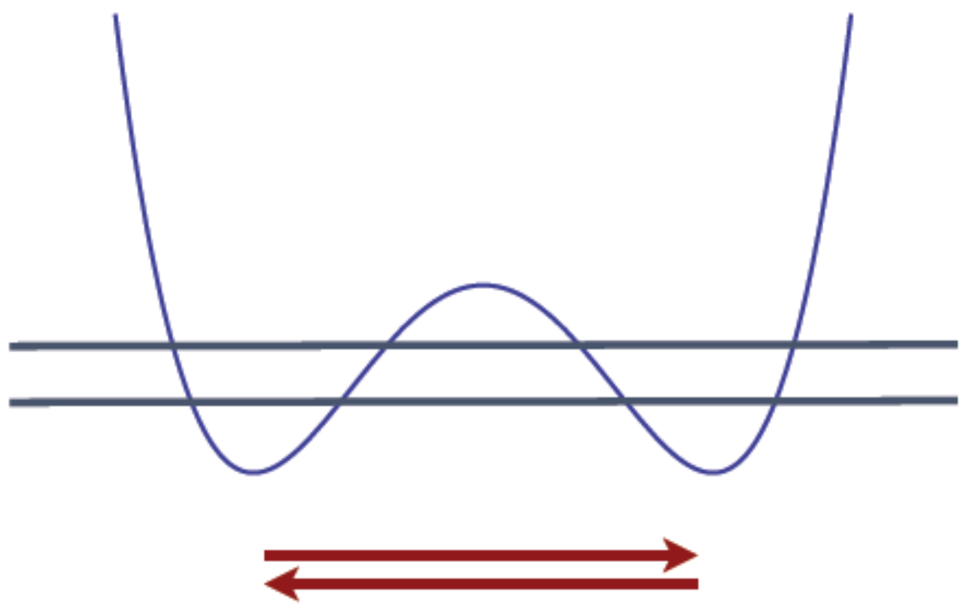
$$a_n \sim (2n)!$$

Borel singularities on the positive Borel axis.

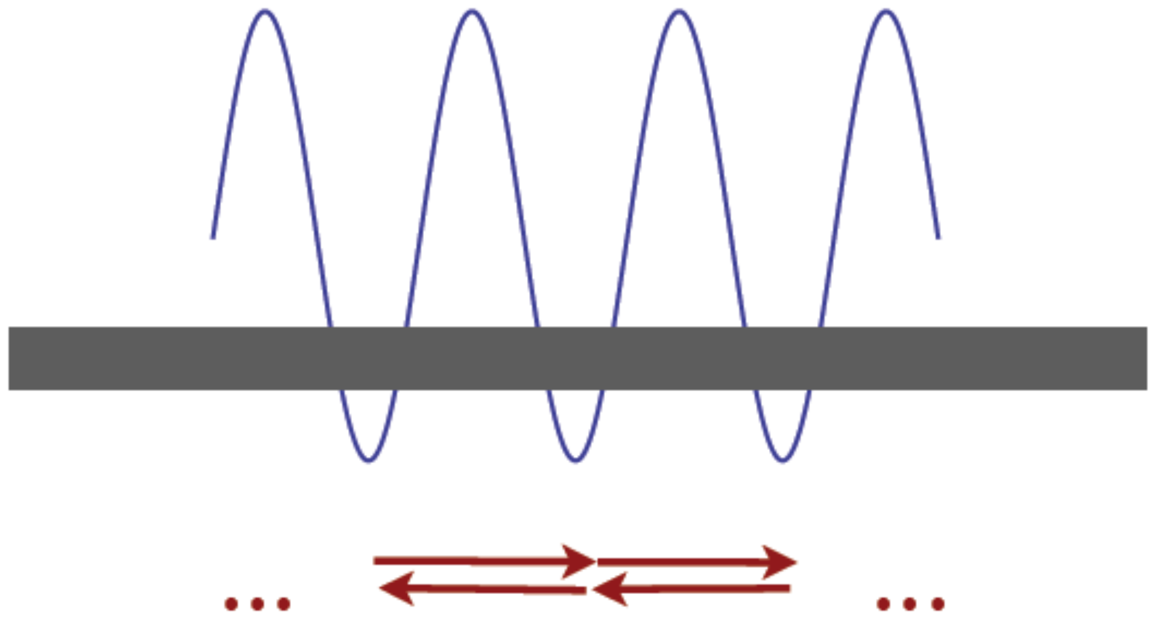
physics: Electric field causes (real) energy level shifts

and ionization (imaginary, exponentially small)

Instantons and Non-Perturbative Physics



(phase transitions)



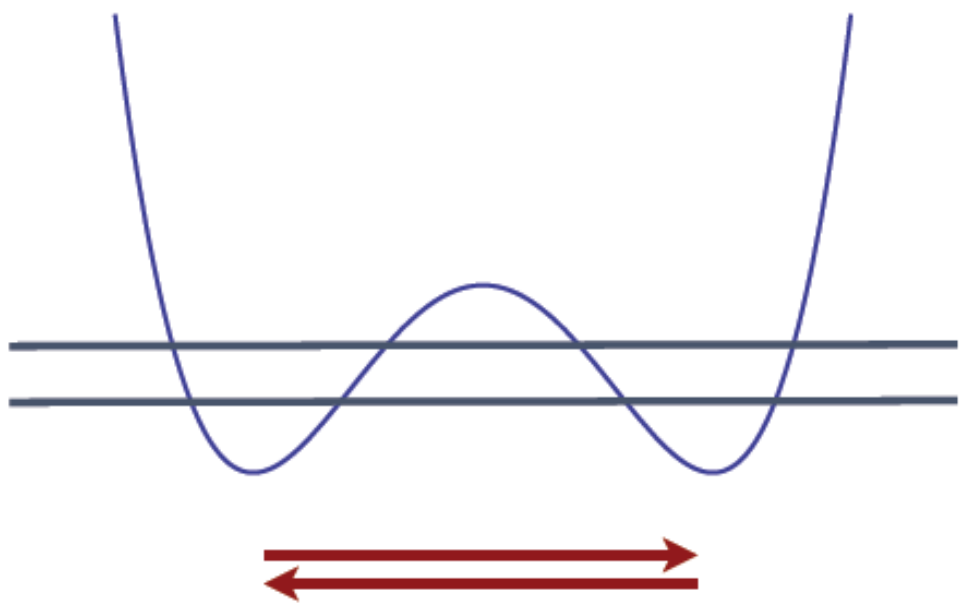
(band structure)

- exponentially small non-perturbative splitting due to tunneling
- Yang-Mills theory and QCD have aspects of both systems

less familiar: perturbation theory is non-alternating divergent

but these systems are stable ???

Instantons and Non-Perturbative Physics



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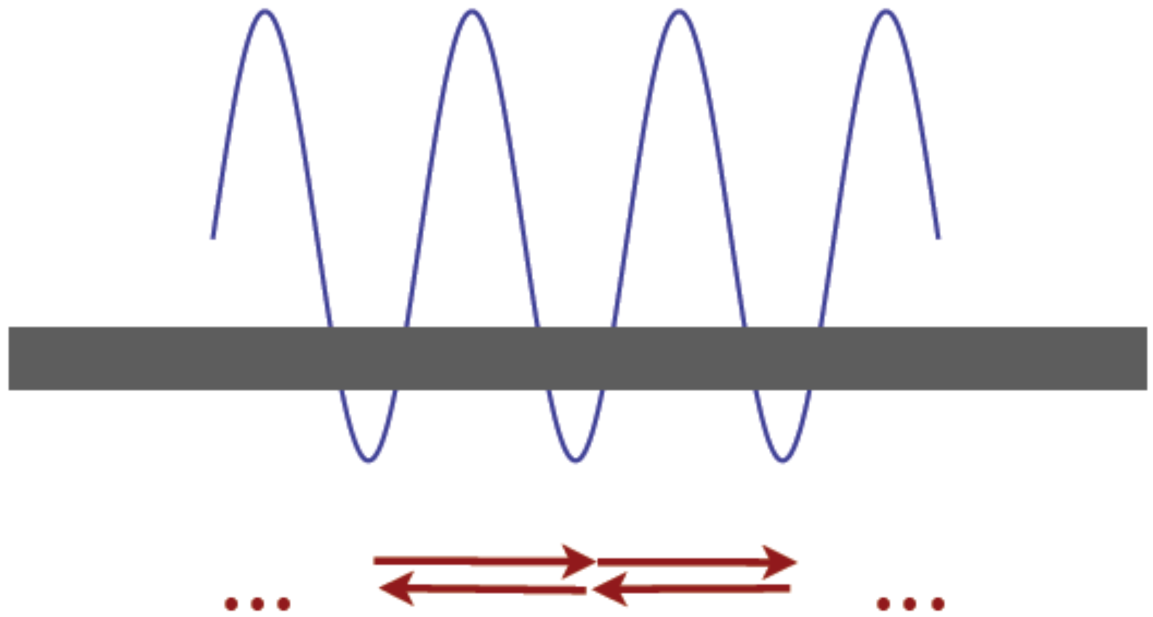

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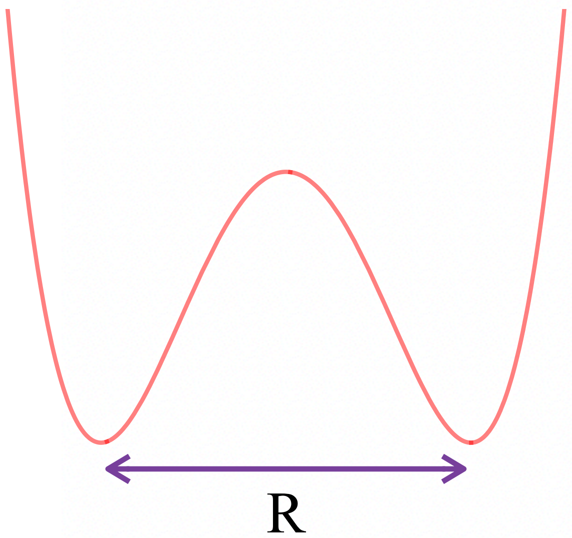
• resolution: trans-series encodes cancellations between imaginary terms

Bogomolny, Zinn-Justin, … ~1980

VOLUME 52, NUMBER 13 PHYSICAL REVIEW LETTERS 26 MARCH 1984

$1/R$ Expansion for H_2^+: Analyticity, Summability, Asymptotics, and Calculation of Exponentially Small Terms



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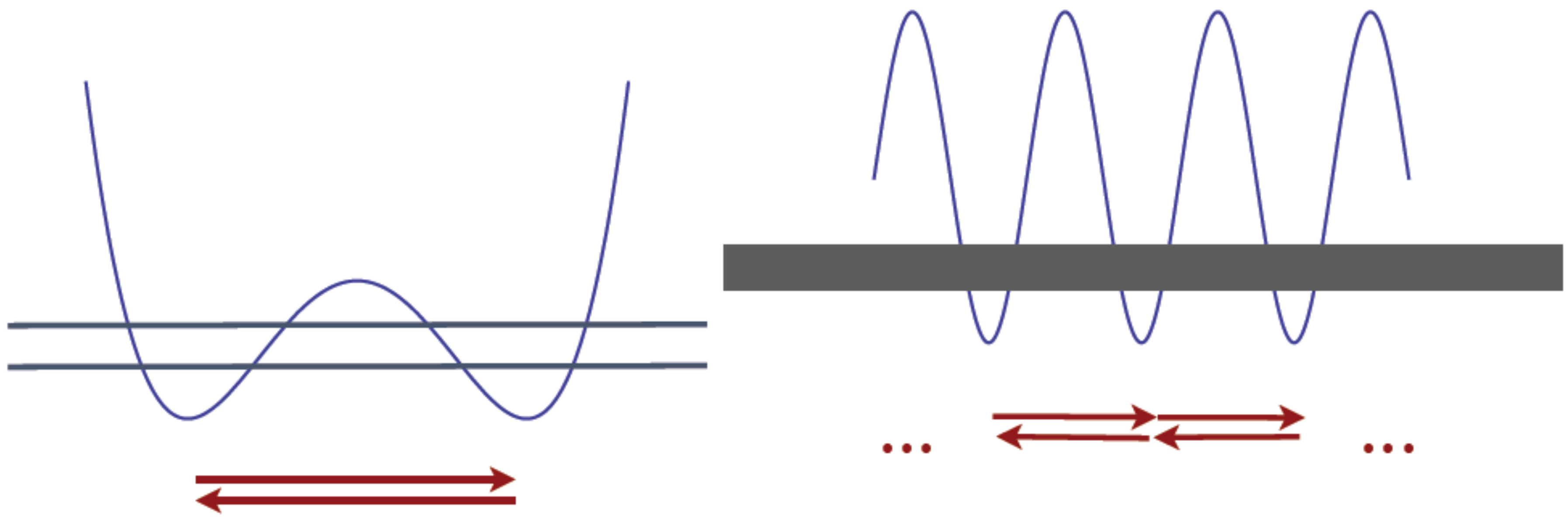
(Received 8 November 1983)

The $1/R$ perturbation series for H_2^+ has a complex Borel sum whose imaginary part determines the asymptotics of the perturbed energy coefficients $E^{(N)}$. The full asymptotic expansion for the energy includes complex, exponentially small terms:

$$E(R) \sim \sum E^{(N)}(2R)^{-N} + e^{-R/n}\sum a^{(N)}(2R)^{-N} + e^{-2R/n}\Big[\sum d^{(N)}(2R)^{-N} + \log R \text{ terms}\Big] \pm ie^{-2R/n}\sum c^{(N)}(2R)^{-N} + \dots .$$

The explicit imaginary terms cancel the implicit imaginary part of the Borel sum. An exact relation between the double-well gap series, $\exp(-R/n)\sum a^{(N)}(2R)^{-N}$, and the $i\exp(-2R/n)$ series is derived.

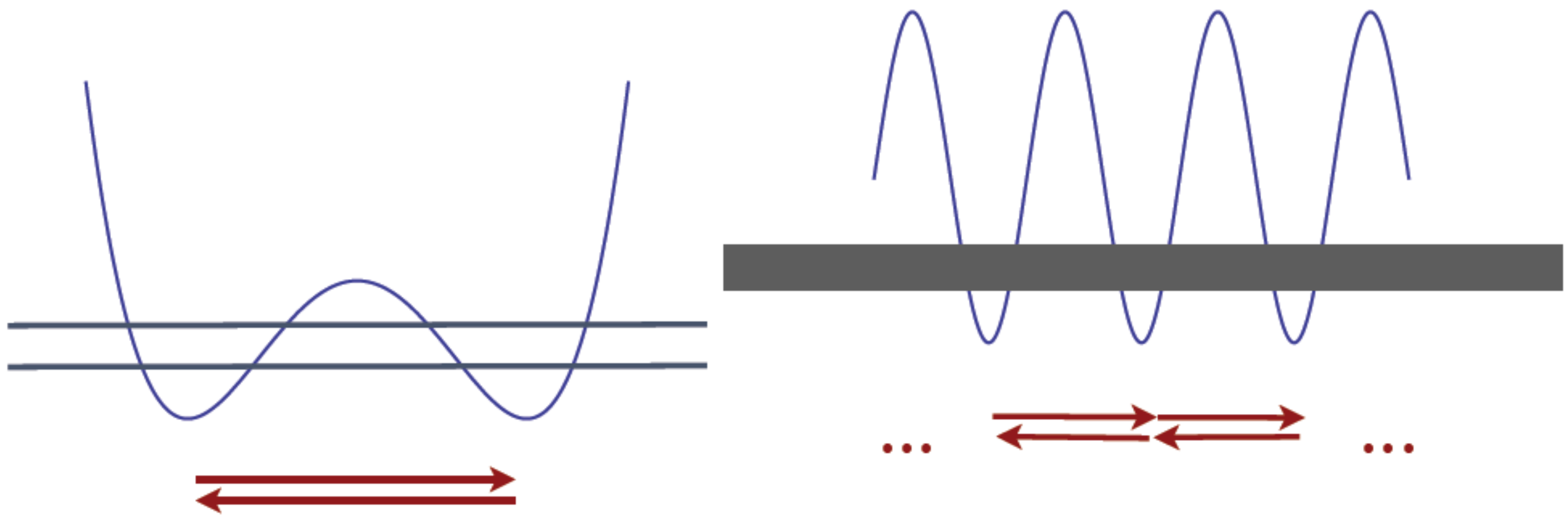
Resurgence in Quantum Mechanical Instanton Models



- trans-series for energy, including non-perturbative splitting:

$$E_\pm(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right] \mathcal{P}_{\rm inst}(\hbar, N) + \ldots$$

Resurgence in Quantum Mechanical Instanton Models



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- fluctuations about first non-trivial saddle:

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

perturbation theory encodes everything … to all orders … in all regions

Alvarez/Casares 2000, Alvarez 2004; GD/Unsal 2014, …

Resurgence in QM

resurgent relations in QM path integrals with an infinite number of saddles

Resurgence and Phase Transitions: Multi-Parameter Trans-Series

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right)$$

- in general, we are interested in many parameters

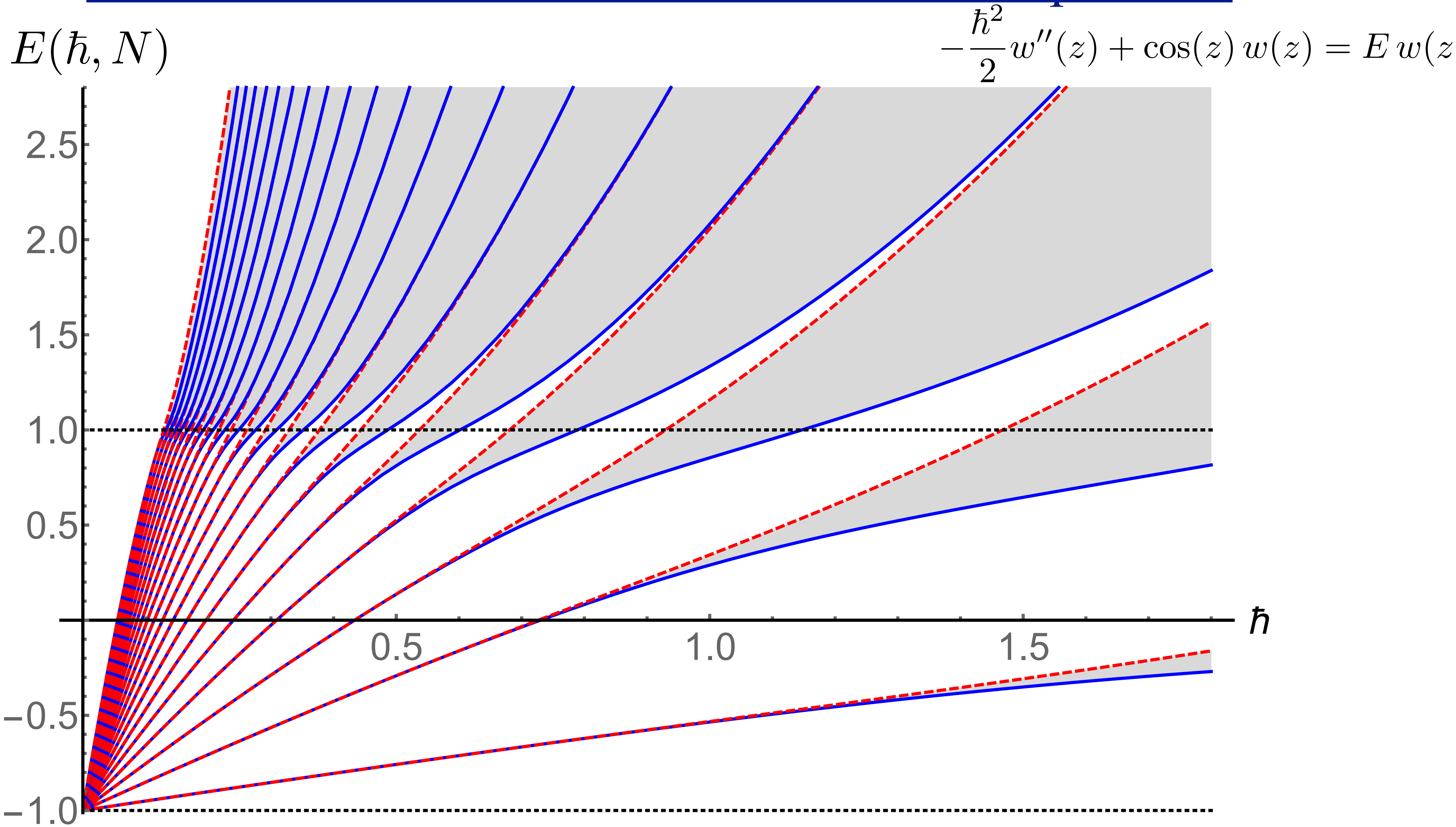
$$Z(\hbar) \;\to\; Z(\hbar, \text{masses}, \text{couplings}, \mu, T, B, \ldots)$$

- e.g., for a phase transition: large N ``thermodynamic limit''

$$Z(\hbar) \;\to\; Z(\hbar, N), \text{ and } N \to \infty$$

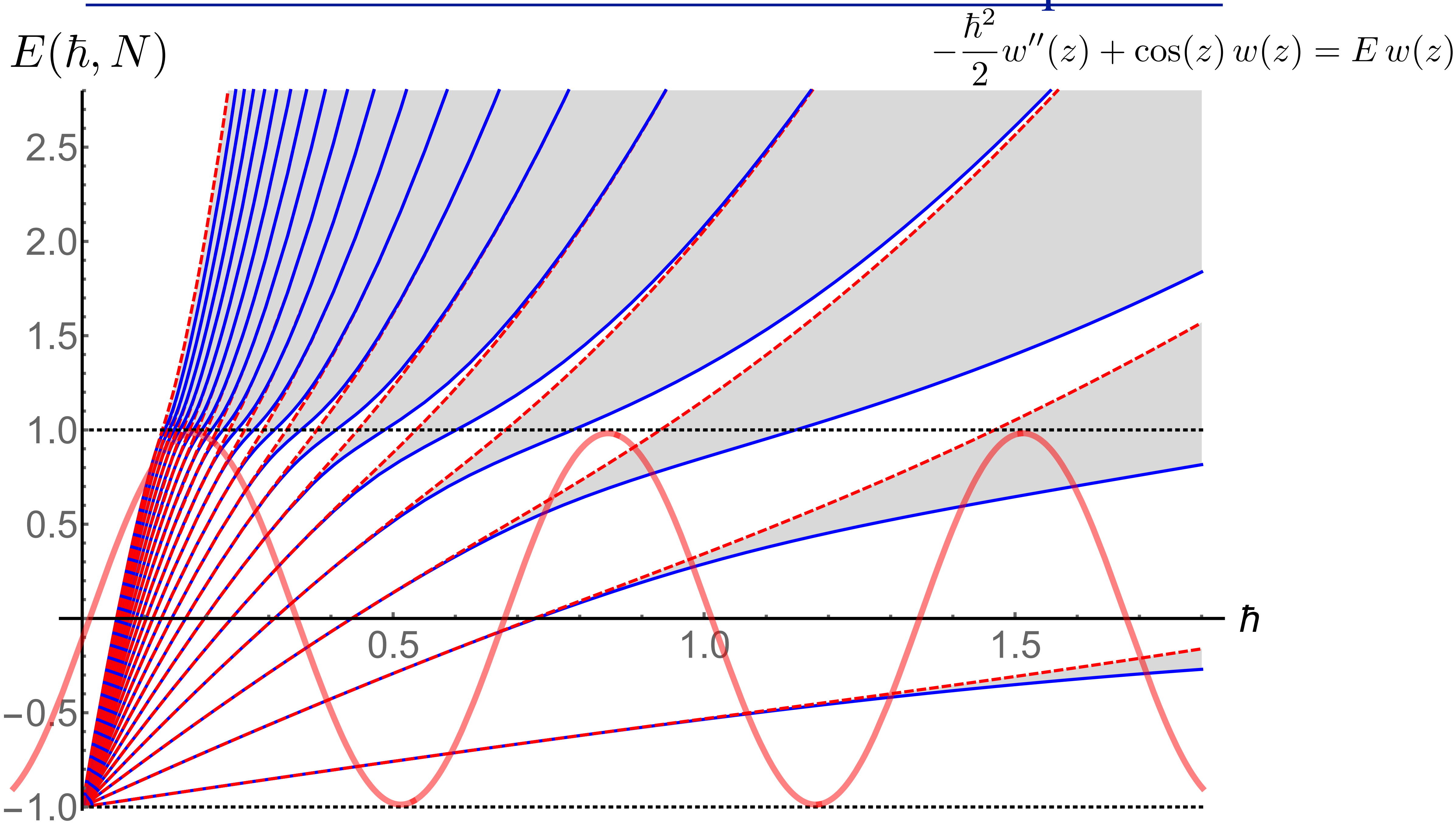
- multiple parameters: different limits are possible
- "uniform" 't Hooft limit: $N \to \infty \;,\; \hbar \to 0 \;:\; \hbar\, N = \text{fixed}$
- trans-series transmutes into a very different form in the large N limit
- hallmark of a Stokes transition

Phase Transition in the Periodic Potential Spectrum



- N = band/gap label; $\hbar$ = coupling
- phase transition: narrow bands vs. narrow gaps: $\hbar N = \frac{8}{\pi}$
- real instantons vs. complex instantons
- phase transition = "instanton condensation"
- universal Stokes transition

Phase Transition in the Periodic Potential Spectrum



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- <u>universal</u> Stokes transition

Resurgence in QFT: Euler-Heisenberg Effective Action

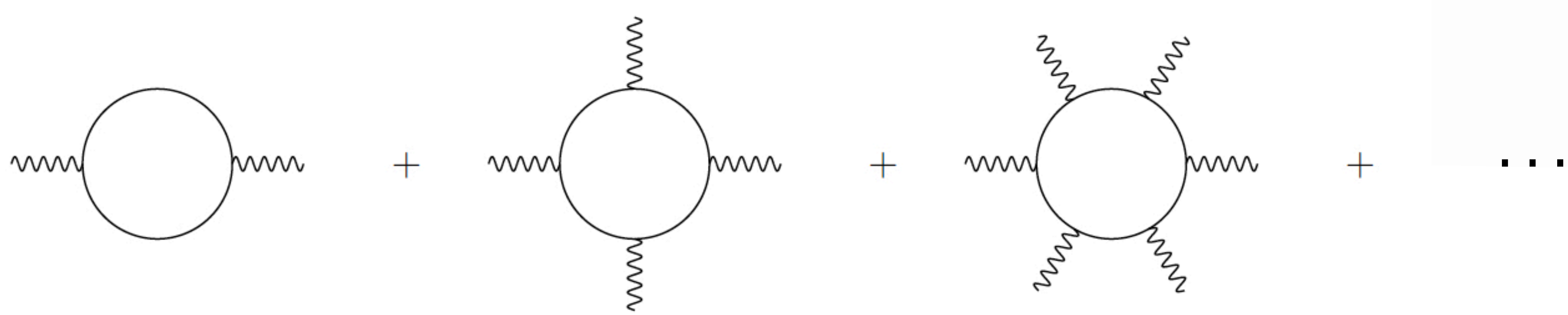
Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{h c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$



- paradigm of effective field theory
- integral representation = Borel sum
- analogue of Stark ionization and Dyson's argument: "Schwinger effect"

$$\mathrm{Im}[\mathcal{L}] \sim e^{-m^2\pi/(e\mathcal{E})}$$

Stokes Phase Transition in QFT

- "Schwinger effect" with *monochromatic* E field: $E(t) = \mathcal{E}\, \cos(\omega\, t)$
- Keldysh adiabaticity parameter: $\gamma \equiv \frac{m\, c\, \omega}{e\, \mathcal{E}}$
- WKB: $\Gamma_{\rm QED} \sim \exp\left[-\pi \frac{m^2\, c^3}{e\, \hbar\, \mathcal{E}}\, g(\gamma)\right]$

Keldysh, 1964;
Brezin/Itzykson, 1970;
Popov, 1971

$$\Gamma_{\rm QED} \sim \begin{cases} \exp\left[-\pi \frac{m^2\, c^3}{e\, \hbar\, \mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(tunneling)} \\ \left(\frac{e\, \mathcal{E}}{m\, c\, \omega}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(multiphoton)} \end{cases}$$

- phase transition: tunneling vs. multi-photon "ionization"
- phase transition: real vs. complex instantons (GD, Dumlu,1004.2509, 1102.2899)
- similar to the transition for the QM cosine potential
- Borel transform is no longer meromorphic
- SLAC (Snowmass LoI) & DESY experiments aim to probe the transition region

World-line Instantons for Intense Field Physics

Feynman worldline representation for one-loop effective action — Feynman 1949, 1951

$$\Gamma[A] = -\int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int \mathcal{D}x(\tau) e^{-\int_0^T d\tau \left(\dot{x}_\mu^2 + A_\mu(x)\dot{x}_\mu\right)}$$

O. Alvarez/Affleck/Manton;
GD, Schubert

- double saddle-point approximation (cf. Gutzwiller)

$$\ddot{x}_\mu = F_{\mu\nu}(x)\dot{x}_\nu$$ ⟶ closed loop with action = S(T, params)

$$\frac{\partial S(T,\text{params})}{\partial T} = -m^2$$ ⟶ T saddle action = S(m^2, params)

- localized intense fields involve complex saddles of the path integral
- particle production = Stokes phenomenon
- interference effects can lead to substantial (exponential) enhancement
- efficient approach to the quantum control problem
- improved semiclassical methods for scattering processes in intense fields

Resurgence and Large N Phase Transitions in Matrix Models

2d lattice Yang-Mills: Gross-Witten-Wadia unitary matrix model

$$Z(t,N)=\int_{U(N)} DU\, \exp\left[\frac{N}{t}\mathrm{tr}\left(U+U^\dagger\right)\right]$$

Gross-Witten, 1980
Wadia, 1980
Marino, 2008

Z depends on two parameters: 't Hooft coupling t, and matrix size N

$$Z(t,N)=\det\left[I_{j-k}\left(\frac{N}{t}\right)\right]_{j,k=1,\dots N}$$

3rd order phase transition in the "thermodynamic" large N limit

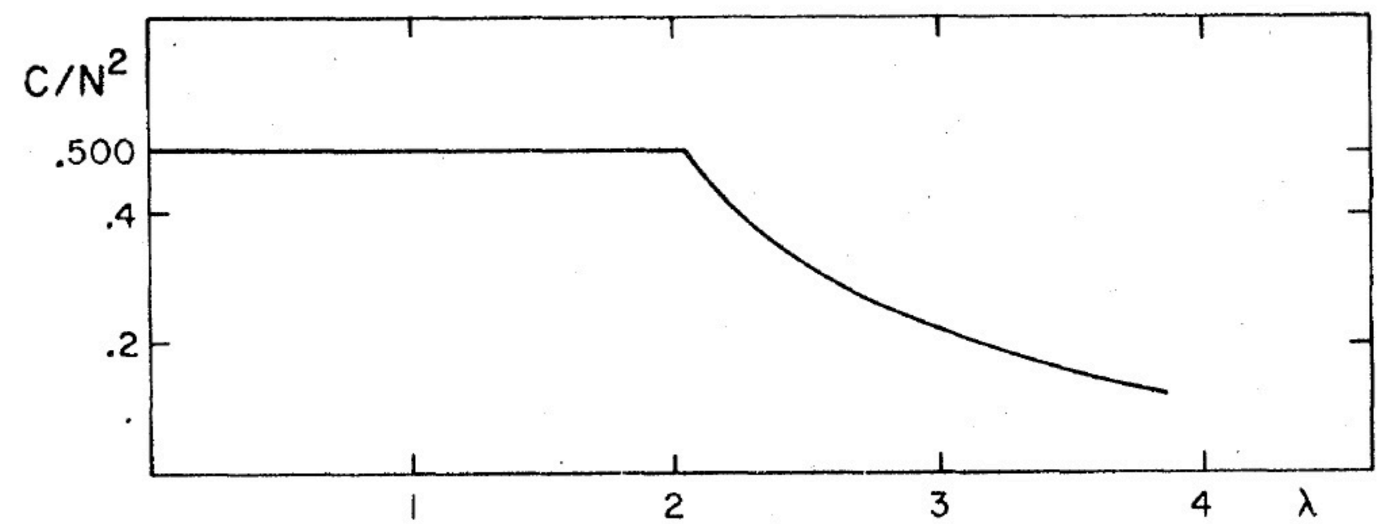


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

"order parameter" $\Delta(t,N)\equiv\langle\det U\rangle$ $\Rightarrow$ physical observables

$$t^2\Delta''+t\Delta'+\frac{N^2\Delta}{t^2}\left(1-\Delta^2\right)=\frac{\Delta}{1-\Delta^2}\left(N^2-t^2\left(\Delta'\right)^2\right)$$

P. Rossi 1982

nonlinear ODE: ideal for resurgent analysis

Resurgence in Weak Coupling Large N Trans-Series

ODE $\Rightarrow$ large N weak coupling trans-series:

Ahmed, GD, 2017

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} + \frac{\sigma_{\rm weak}\, e^{-N S_{\rm weak}(t)}}{\sqrt{4\pi N S'_{\rm weak}(t)}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

weak coupling large N action:

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,{\rm arctanh}\left(\sqrt{1-t}\right)$$

“one-instanton" fluctuations: coefficients are functions of t

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{1}{N} + \dots$$

Resurgence in Weak Coupling Large N Trans-Series

ODE $\Rightarrow$ large N weak coupling trans-series: Ahmed, GD, 2017

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resurgence: large-order growth of “perturbative coefficients”:

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\frac{\Gamma(2n-\frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right]$$

Resurgence in Strong Coupling Large N Trans-Series

- large N strong-coupling: $\Delta(t,N) \approx \sigma\, J_N\left(\frac{N}{t}\right)$

- Debye expansion: completely different trans-series

$$\Delta(t,N) \sim \frac{t\, e^{-N S_{\rm strong}(t)}}{\sqrt{2\pi N |S'_{\rm strong}(t)|}} \sum_{n=0}^\infty \frac{U_n^{(0)}(t)}{N^n} + \frac{1}{4(t^2-1)}\left(\frac{t\, e^{-N S_{\rm strong}(t)}}{\sqrt{2\pi N |S'_{\rm strong}(t)|}}\right)^3 \sum_{n=0}^\infty \frac{U_n^{(1)}(t)}{N^n} + \dots$$

- large N strong-coupling action: $S_{\rm strong}(t) = {\rm arccosh}(t) - \sqrt{1-\frac{1}{t^2}}$

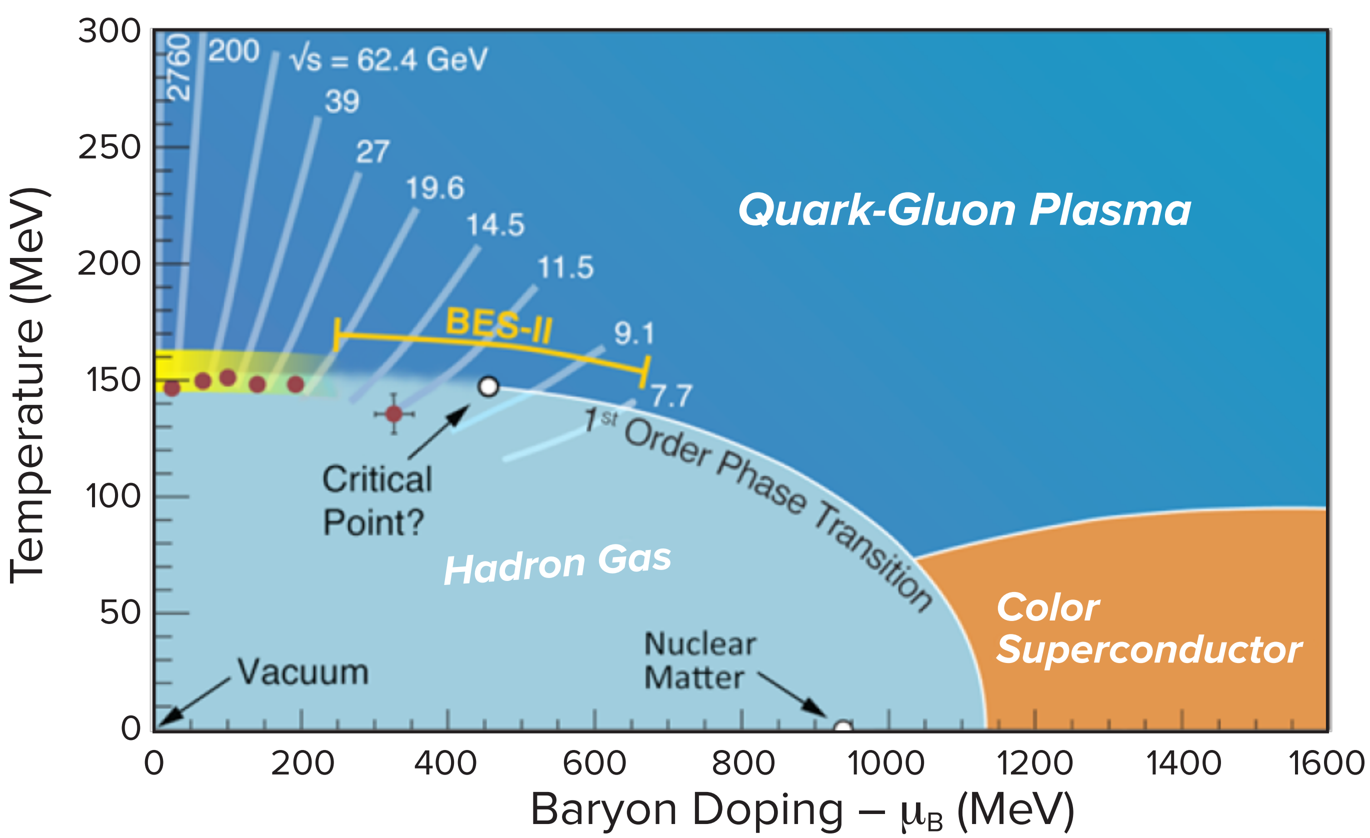
- low-order/large-order resurgence relation (for all t):

$$U_n(t) \sim \frac{(-1)^n (n-1)!}{2\pi (2 S_{\rm strong}(t))^n}\left(1 + U_1(t)\frac{(2S_{\rm strong}(t))}{(n-1)} + U_2(t)\frac{(2S_{\rm strong}(t))^2}{(n-1)(n-2)} + \dots\right)$$

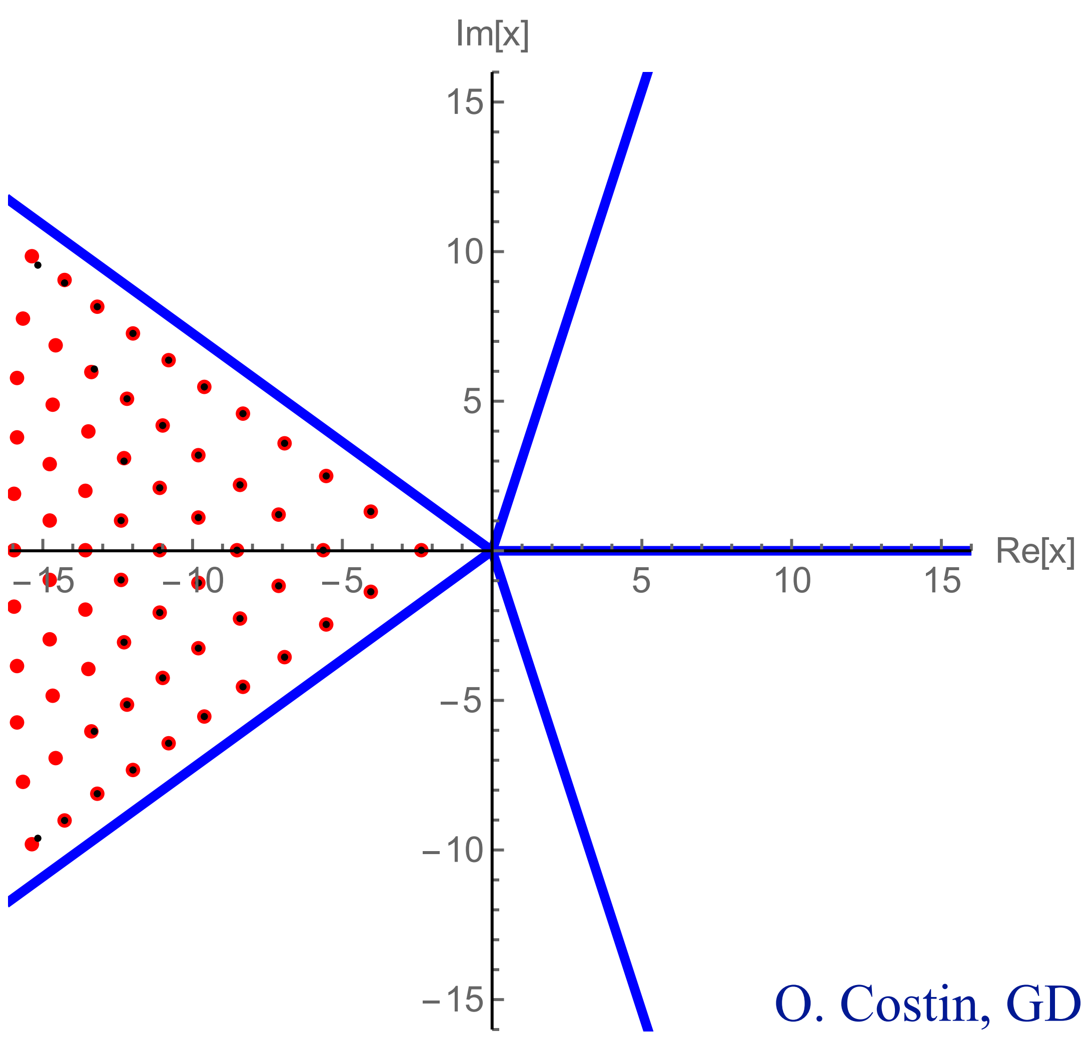
Resurgent Extrapolation

O. Costin, GD: 1904.11593, 2003.07451
2009.01962, 2108.00145

- sometimes perturbation theory/asymptotics is the ONLY thing we can do
- resurgence implies perturbation theory encodes non-perturbative physics
- practical question: how much global non-perturbative information can be decoded from a FINITE number of perturbative coefficients ?



Basar, GD, Yin: 2112.14269



O. Costin, GD: 1904.11593

tritronquee of Painleve I eqn.

Resurgent Extrapolation: Euler-Heisenberg at 1-loop

GD, Z. Harris: 2101.10409

$$\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) = -\frac{B^2}{2}\int_0^\infty \frac{dt}{t^2}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right) e^{-m^2 t/(eB)}$$

$$\sim \frac{B^2}{\pi^2}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^\infty (-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}}\zeta(2n+4)\left(\frac{eB}{m^2}\right)^{2n} \quad , \quad eB \ll m^2$$

$$\sim \frac{1}{3}\cdot\frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma + \frac{6}{\pi^2}\zeta'(2)\right) + \dots \quad , \quad eB \gg m^2$$

- weak to strong B field extrapolation ?
- B field to E field analytic continuation ?

Resurgent Extrapolation: Euler-Heisenberg at 1-loop

GD, Z. Harris: 2101.10409

$$
\begin{aligned}
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\end{aligned}
$$

- weak to strong B field extrapolation ?
- B field to E field analytic continuation ?

start with just 10 terms

Resurgent Extrapolation: Euler-Heisenberg at 1-loop

GD, Z. Harris: 2101.10409

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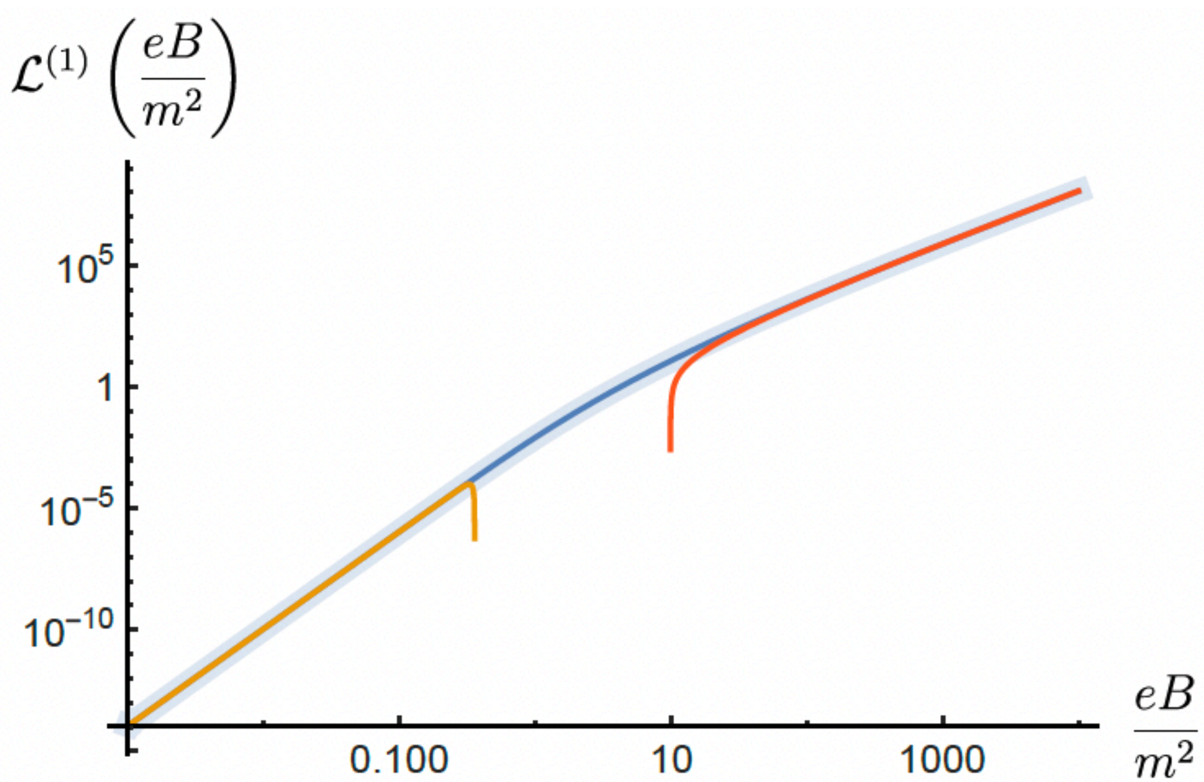
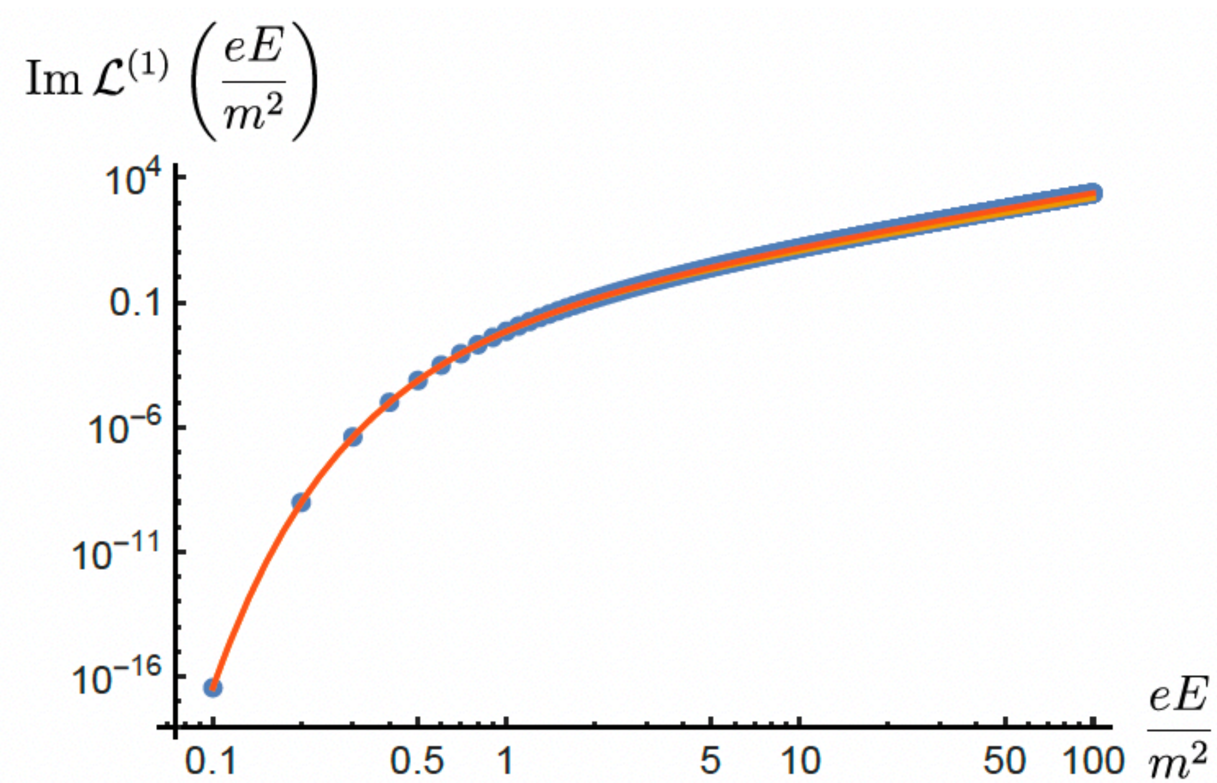




- accurate over many orders of magnitude (from just 10 input terms!)

Resurgent Extrapolation: Euler-Heisenberg at 2-loop

GD, Z. Harris: 2101.10409

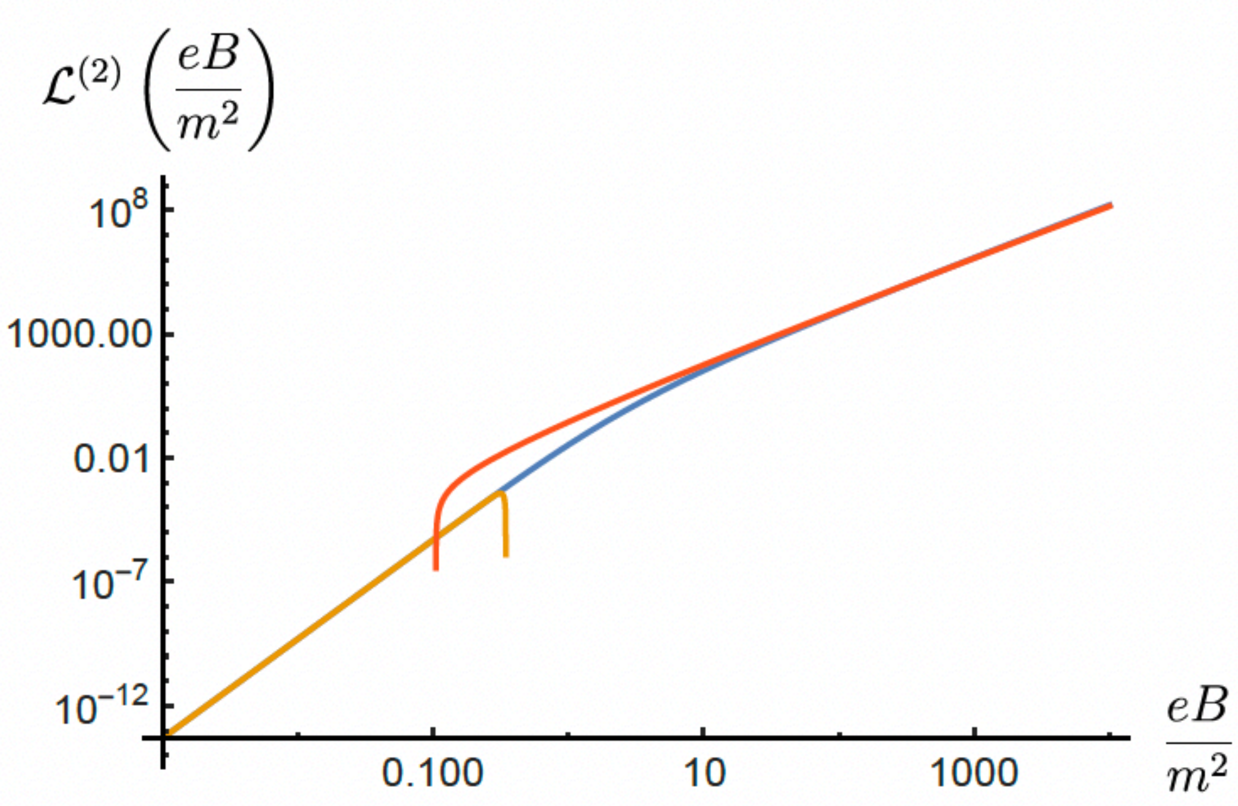
- 2 loop: Ritus double-integral representation

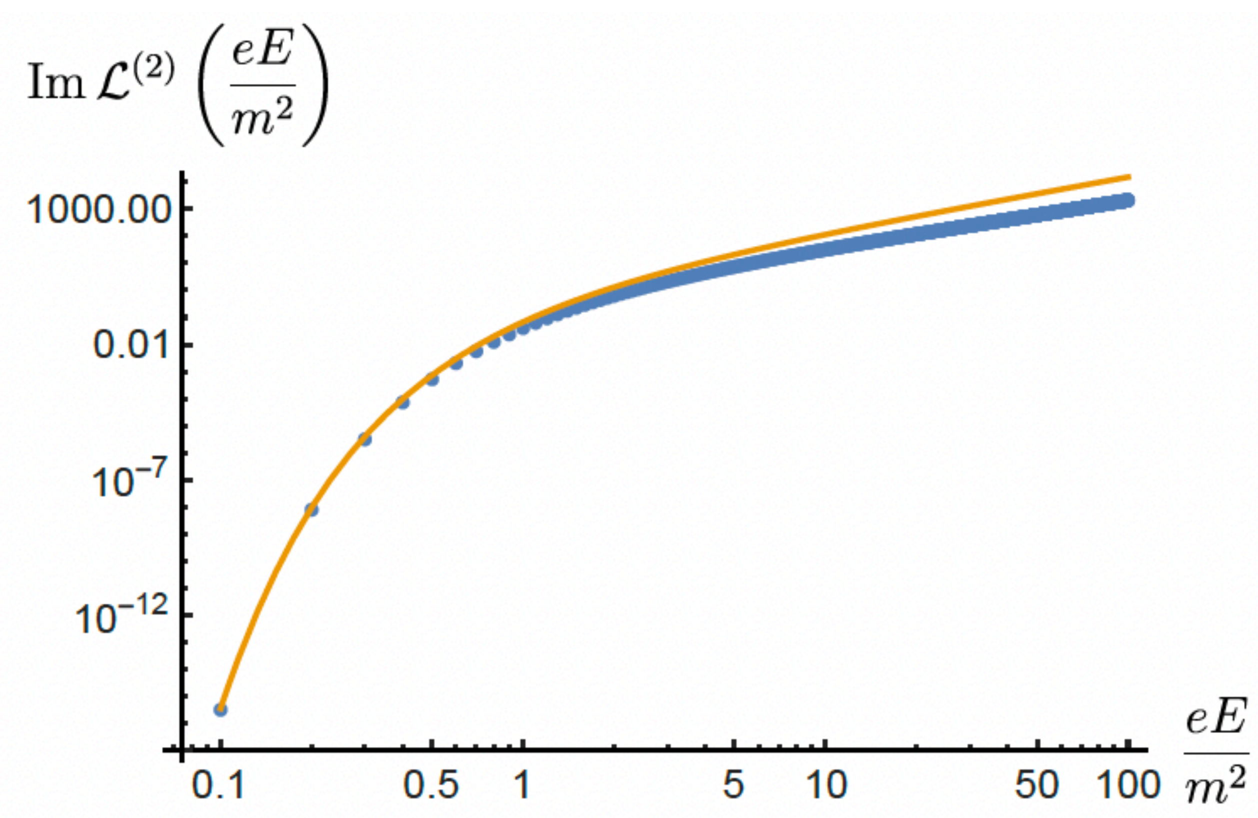
$$\mathcal{L}^{(2)}\left(\frac{eB}{m^2}\right) \sim \frac{B^2}{\pi^2}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{eB}{m^2}\right)^{2n} \quad , \quad eB \ll m^2$$

$$\sim \frac{1}{4}\cdot\frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma - \frac{5}{6} + 4\zeta(3)\right) + \dots \quad , \quad eB \gg m^2$$

- weak to strong *B* field extrapolation ?
- *B* field to *E* field analytic continuation ?

start with just 10 terms





- accurate over many orders of magnitude (from just 10 input terms!)

Resurgent Extrapolation: new approach to intense-field problems

• in progress: (GD, Z. Harris, to appear)

  • application to field inhomogeneities
  • application to strong gravitational backgrounds

• don't struggle with exact integral representations
• instead: use perturbative expansion from the start
• then decode the non-perturbative effects using resurgence

• resurgence suggests that the non-perturbative physics is encoded in perturbation theory

• key question: how much perturbative information do we need in order to reliably reconstruct the non-perturbative physics ?
• new practical decoding procedures

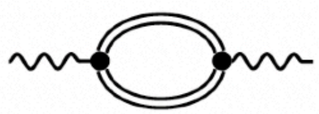
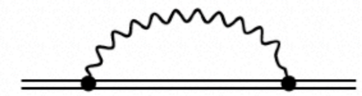
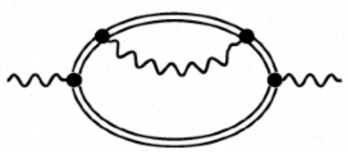
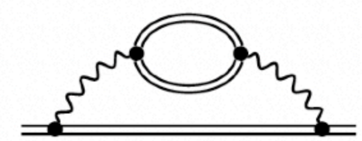
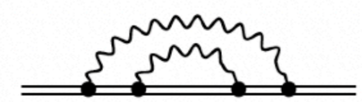
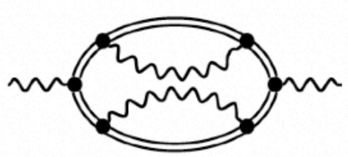
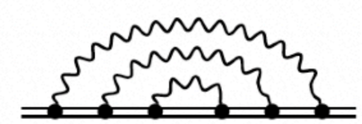
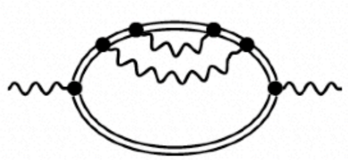
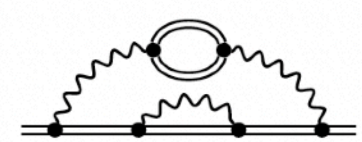
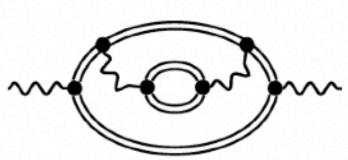
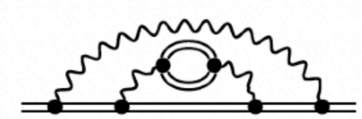
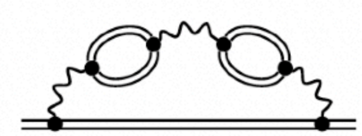
O. Costin, GD: 2009.01962, 2108.00145

Non-perturbative QED: Probing the Ritus-Narozhnyi Conjecture

- QED expansion parameter in a constant crossed field, for $\chi \gg 1$, is $\alpha\chi^{2/3}$

$$\chi \equiv e\sqrt{-(F_{\mu\nu}p^\nu)^2}/m^3$$

Ritus, Narozhnyi: 1970s/80s

			1 loop				
(1a)		$\alpha\boldsymbol{\chi}^{\mathbf{2/3}}$	[12]	(1b)		$\alpha\boldsymbol{\chi}^{\mathbf{2/3}}$	[13]
			2 loops				
(2a)		$\alpha^2\boldsymbol{\chi}^{\mathbf{2/3}}\log\chi$	[16]	(2b)		$\alpha^2\boldsymbol{\chi}\log\chi$	[14, 21]
				(2c)		$\alpha^2\chi^{2/3}\log\chi$	[15]
			3 loops				
(3a)		$\alpha^3\chi^{2/3}\log\chi$	[17]	(3d)		$\alpha^3\chi^{2/3}\log^2\chi$	[17]
(3b)		$\alpha^3\chi^{2/3}\log\chi$	[17]	(3e)		$\alpha^3\chi^{4/3}$	[17]
(3c)		$\alpha^3\boldsymbol{\chi}\log^2\chi$	[18]	(3f)		$\alpha^3\chi\log^2\chi$	[18]
				(3g)		$\alpha^3\boldsymbol{\chi}^{\mathbf{5/3}}$	[18]

- sophisticated analysis of multi-parameter integrals … complicated !

Mironov et al: 2003.06909; A. Fedotov: 1608.02261; Di Piazza et al 2006.15370; Torgrimsson 2007.08492; Heinzl et al 2101.12111, …

Conclusions

- "resurgence" is based on a new and improved form of asymptotics
- deep(er) connections between perturbative and non-perturbative physics
- recent applications to differential eqs, QM, QFT, string theory, …
- resurgent extrapolation: high-precision extraction of physical information from finite order expansions
- outlook: computational access to strongly-coupled systems, finite density, phase transitions, particle production, far-from-equilibrium physics, …

Further topics not covered today ….

- Lefschetz thimbles and bions
- "Exact WKB"
- Chern-Simons theory
- Non-perturbative effective actions
- Yang-Mills & QCD
- Dualities and Modularity
- Integrability and large N
- Renormalons and the OPE
- Hopf algebraic renormalization