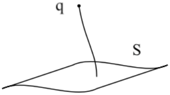
Generalizations of classical relativity theorems

Eleni-Alexandra Kontou
TPPC

3 May 2023

For students: focal points and how to find them

Focal points



Timelike geodesic normal to spacelike hypersurface S

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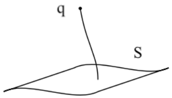

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A geodesic issuing normally from a spacelike hypersurface S and is continued past a focal point no longer locally extremizes length.

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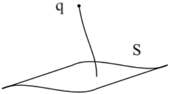

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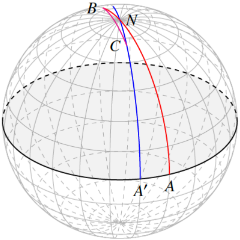

- N: focal point
- AN equal to $A'N$
- ANB longer than $A'CB$

The index form

Problem

Maximise proper time among timelike curves joining surface S to point q.

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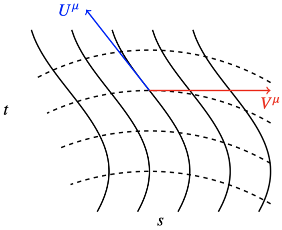
Consider an 1-parameter family of smooth curves $\gamma_s : [0, \tau] \to M$

$$U^\mu = \frac{\partial \gamma_s^\mu}{\partial t}, \qquad V^\mu = \frac{\partial \gamma_s^\mu}{\partial s}$$



$$L[\gamma] = \int_0^\tau |\dot{\gamma}(t)|\, dt, \qquad |\dot{\gamma}(t)| = \sqrt{g_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu}$$

The index form

The first variation of length

$$\left.\frac{dL[\gamma_s]}{ds}\right|_{s=0} = 0$$

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The second variation of length (index form)

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Focal point test

$I[V] \geq 0$ for some $V^\mu \implies \exists$ focal point in $(0, \tau]$

Null geodesics

Problem 1 L does not vary smoothly for all causal curves

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Energy or action integral

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The Hessian

$$\mathbf{H}[V] \equiv \left.\frac{d^2 E[\gamma_s]}{ds^2}\right|_{s=0}$$

Problem 2 How do we measure length? Fix parametrization on the hypersurface

Generalizations of classical relativity theorems

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Outline

Based on: 1907.13604, 2012.11569 and 2303.06788

Introduction

Motivation

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- These assumptions are usually violated in the semiclassical regime: quantum field effects become important

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- These assumptions are usually violated in the semiclassical regime: quantum field effects become important

Question

Which classical relativity theorems still hold in the semiclassical regime and what are the new assumptions?

The theorems

Singularity theorems

Predict the existence of singularities: a spacetime is singular if it possesses at least one incomplete geodesic.

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$$\boxed{\text{Hawking area theorem}} \longleftrightarrow \text{Tension} \longleftrightarrow \boxed{\text{Hawking radiation}}$$

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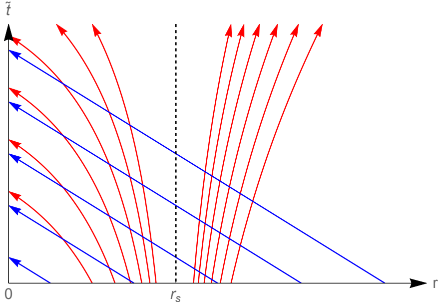
$$\boxed{\text{Hawking area theorem}} \longleftrightarrow \text{Tension} \longleftrightarrow \boxed{\text{Hawking radiation}}$$

- Is the area theorem always violated semiclassically? Why?
- Is there a weaker version of the theorem obeyed by quantum fields?

The classical theorems

Singularity theorem structure

1. **The initial or boundary condition**
   There exists a trapped surface (null geodesics)



Singularity theorem structure

2. **The energy condition**
   Restriction on the stress-energy tensor expressing "physical" properties of matter.
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Proof structure

1. Initial condition: Geodesics start focusing
2. Energy condition: Focusing continues
3. Causality condition: No focal points

$\Rightarrow$ Geodesic incompleteness

The area theorem structure

1. **Causality condition**
   The spacetime is strongly asymptotically predictable:
   - No naked singularities
   - Asymptotically flat
   - There is an area with Cauchy surfaces

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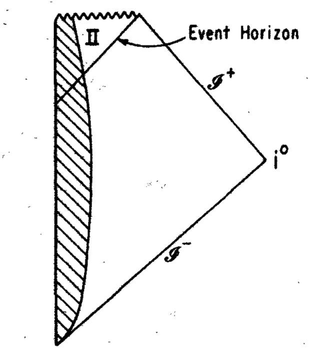

Black hole: Area II (no null infinity)
Event horizon: the boundary

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2. **The energy condition**
   The null convergence condition: $R_{\mu\nu} U^\mu U^\nu \geq 0$

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   The null convergence condition: $R_{\mu\nu} U^{\mu} U^{\nu} \geq 0$

Proof structure

1. Assume the area of the horizon decreases: the null geodesics normal to the horizon are focusing
2. Energy condition: Focusing continues
3. Causality condition: No focal points

$\Rightarrow$ The area of the horizon cannot decrease

Focal points

Definition

Let P be a co-dimension 2 spacelike submanifold and let γ be a causal geodesic normal to P. Then a focal point on γ is a point where the causal geodesic no longer extremizes the action integral.

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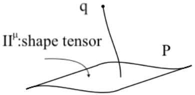

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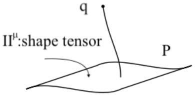

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First derivative

$$\left.\frac{dE[\gamma_s]}{ds}\right|_{s=0} = 0$$

The Hessian

$$\mathbf{H}[V] \equiv \left.\frac{d^2 E[\gamma_s]}{ds^2}\right|_{s=0}$$

Focusing theorem

$$\mathbf{H}[V] = \int_0^\ell \left[(\nabla_U V_\mu)(\nabla_U V^\mu) + R_{\mu\nu\alpha\beta} \overbrace{U^\mu}^{\text{tangent}} \underbrace{V^\nu}_{\text{variation}} V^\alpha U^\beta\right] d\lambda - U_\mu \mathrm{I\!I}^\mu(V,V)\Big|_{\gamma(0)}$$

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Let e_i with $i = 1, \ldots, n-2$ be a tetrad basis on P, and parallel transport them along γ to generate $\{E_i\}_{i=1,\ldots,n-2}$. Then, take f a smooth function with $f(0) = 1$ and $f(\ell) = 0$ and sum over i

$$\sum_{i=1}^{n-2} \mathbf{H}(fE_i, fE_i) = -\int_0^\ell \Big((n-2) f'^2(\lambda) - f^2 R_{\mu\nu} U^\mu U^\nu \Big)\, d\lambda - (n-2) f^2 U_\mu \mathrm{H}^\mu \Big|_{\gamma(0)}$$

$H^\mu = 1/(n-2) \sum_{i=1}^{n-2} \mathrm{I\!I}^\mu(E_i, E_i)$ is the mean normal curvature vector field of P

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Condition for the existence of focal points

- No focal point before q: $\mathbf{H}[V] > 0$ for all V
- Focal point before q: $\mathbf{H}[V] < 0$ for some V

The Penrose singularity theorem

Theorem

(i) M is globally hyperbolic with non-compact $\boxed{\text{Cauchy hypersurfaces}}$ and P is a compact achronal smooth spacelike submanifold of M of co-dimension 2

(ii) Let γ be a null geodesic emanating normally from P with tangent vector U^μ and let everywhere on γ

$$\boxed{R_{\mu\nu}U^\mu U^\nu \geq 0}$$

(iii) For the mean normal curvature vector field $H^\mu = H\hat{H}^\mu$ we have $\boxed{H < 0}$

$\Rightarrow$ Then γ is incomplete and has length less than $\ell = 1/|H|$ where the affine parameter is fixed by requiring $H^\mu d\gamma_\mu/d\lambda = 1$ on P.

Proof

Condition for the formation of a focal point

$$\int_0^{\ell} \left((n-2)f'(\lambda)^2 - f(\lambda)^2 R_{\mu\nu}U^\mu U^\nu\right) d\lambda \leq -(n-2)H$$

Proof

Condition for the formation of a focal point

$$\int_0^\ell \left((n-2)f'(\lambda)^2 - \overbrace{f(\lambda)^2 R_{\mu\nu}U^\mu U^\nu}^{+}\right) d\lambda \leq -(n-2)H$$

- Null convergence condition $R_{\mu\nu}U^\mu U^\nu \geq 0$

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$\Rightarrow$ Focal point for $\ell \geq 1/|H|$

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Condition (i) implies the P-normal null geodesics have no focal points and thus they have length at most ℓ.

The Hawking area theorem

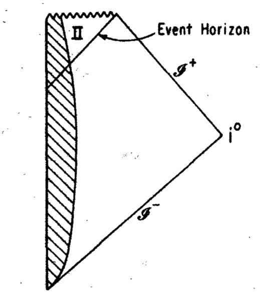
Theorem

(i) M is

$\boxed{\text{strongly asymptotically predictable}}$

(ii) Everywhere on M and for all null vectors U^μ

$$\boxed{R_{\mu\nu} U^\mu U^\nu \geq 0}$$

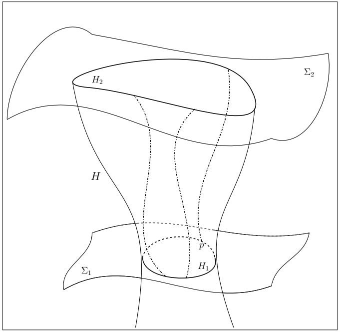


Let Σ_1 and Σ_2 be spacelike Cauchy surfaces for the globally hyperbolic region $\tilde{V}$ such that $\Sigma_2 \subset I^+(\Sigma_1)$, and given H the event horizon we define

$$\mathscr{H}_1 = H \cap \Sigma_1, \qquad \mathscr{H}_2 = H \cap \Sigma_2.$$

Then the area of $\mathscr{H}_2$ is greater or equal than the area of $\mathscr{H}_1$.

Proof



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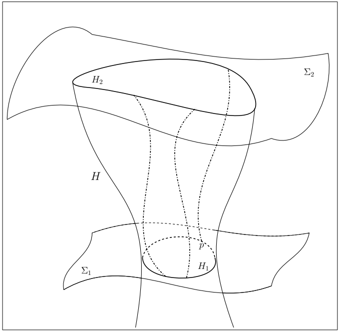
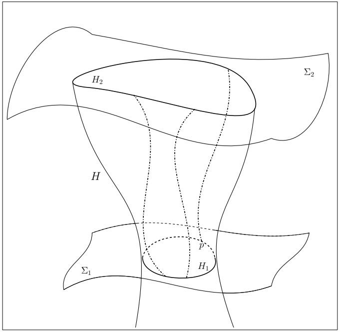


$$\int_0^\ell \Big(\overbrace{(n-2)f'(\lambda)^2}^{(n-2)/\ell} - \overbrace{f(\lambda)^2 R_{\mu\nu} U^\mu U^\nu}^{+} \Big) d\lambda \leq -(n-2) U_\mu H^\mu$$

- Null convergence condition $R_{\mu\nu} U^\mu U^\nu \geq 0$
- Choose a function $f(\lambda) = 1 - \lambda/\ell$
  If $U_\mu H^\mu < 0$ for $\mathscr{H}_1$ we have a focal point for $\ell \geq 1/|U_\mu H^\mu|$.

Proof



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Condition (i) implies there are no focal points on the null generators so $U_\mu H^\mu \geq 0$.

$$\delta_U \mathcal{A}_{\mathscr{H}_1} = \int_{\mathscr{H}_1} \mathrm{H}^\mu U_\mu \geq 0$$

Weakened conditions

Weakened energy conditions

Problem

All pointwise energy conditions are violated by quantum fields

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QFT

Quantum energy inequalities (QEIs) introduce a restriction on the possible magnitude and duration of any negative energy densities or fluxes within a quantum field theory.

Example of a QEI (bound on energy density in Minkowski spacetime)

$$\int dt\, f^2 \langle :T_{\mu\nu} U^\mu U^\nu: \rangle_\omega \geq -\frac{1}{16\pi^2} \int f''(t)^2 dt$$

[Ford, Roman, 1995], [Fewster, Eveson, 1998]

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[Ford, Roman, 1995], [Fewster, Eveson, 1998]

Weakened condition inspired by QEIs

$$\int f(t)^2 R_{\mu\nu} U^\mu U^\nu \, dt \geq -Q_m \|f^{(m)}\|^2 - Q_0 \|f\|^2$$

The rest of the talk

1. Singularity theorems with weakened conditions *inspired* by QEIs
2. A null QEI: the smeared null energy condition
3. Semiclassical theorems and applications: evaporating black holes

Singularity theorems with weaker conditions

Theorem [Fewster, E-AK, 2019]

(i) Energy condition

$$\int_0^\ell f(\lambda)^2 \overbrace{R_{\mu\nu}U^\mu U^\nu}^{\rho} \, d\lambda \geq -Q_m \|f^{(m)}\|^2 - Q_0 \|f\|^2$$

and Scenario 1: $\rho \geq \rho_0$ for $[0, \ell_0]$: NEC obeyed after we measure H
or Scenario 2: $\rho < -\rho_0$ for $[-\ell_0, 0]$: NEC violated before we measure H

(ii) Initial condition: $H \leq -\nu(Q_m, Q_0, \ell_0, \ell, \rho_0)$

(iii) Causality condition: There exists a non-compact Cauchy surface.

$\Rightarrow$ The spacetime is null geodesically incomplete.

Idea of the proof

Condition for the formation of a focal point

$$\int_0^\ell \left((n-2)f'(\lambda)^2 - f(\lambda)^2 R_{\mu\nu}U^\mu U^\nu\right) d\lambda \leq -(n-2)H\,, f(0)=1\,, f(\ell)=0$$

Energy condition

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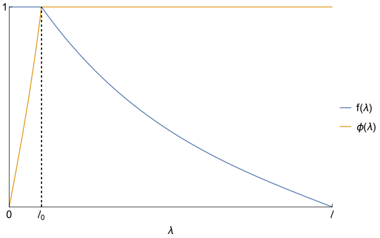
$$\int_0^\ell (\phi(\lambda)f(\lambda))^2 R_{\mu\nu}U^\mu U^\nu\, d\lambda \geq -Q_m \|(\phi f)^{(m)}\|^2 - Q_0\|(\phi f)\|^2\,,\, (\phi f)(0)=(\phi f)(\ell)=0$$



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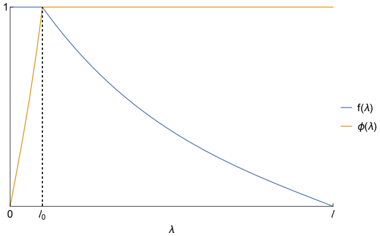

Two options:

- Pick functions for general m
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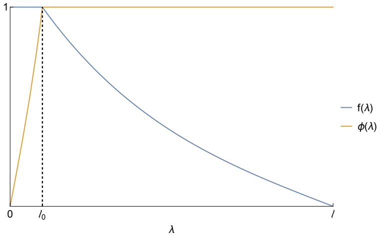

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$$H \leq -\nu(Q_m, Q_0, \ell_0, \ell, \rho_0)$$

Generalized area theorem

Question

What is the weakest energy condition so that the classical area theorem holds?

Generalized area theorem

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Observation

We need to have a focal point for $U_\mu H^\mu < 0$ to conclude the non-decrease of the horizon. Then

$$\int_0^\ell f(\lambda)^2 R_{\mu\nu} U^\mu U^\nu d\lambda \geq (n-2)\|f'\|^2$$

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Example of such a condition: Half averaged null energy condition

$$\int_0^\infty R_{\mu\nu} U^\mu U^\nu d\lambda \geq 0$$

[Lesourd, 2017]

Generalized area theorem

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What happens if we have a condition inspired by QEIs?

Generalized area theorem

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Theorem [E-AK, Sacchi, 2023]

(i) Energy condition

$$\int_0^\ell f(\lambda)^2 \overbrace{R_{\mu\nu} U^\mu U^\nu}^{\rho} \, d\lambda \geq -Q_m \|f^{(m)}\|^2 - Q_0 \|f\|^2$$

and Scenario 1: $\rho \geq \rho_0$ for $[0, \ell_0]$

(ii) Causality condition: M is strongly asymptotically predictable

Then we have a bound on the change of the horizon area

$$\delta_U \mathcal{A}_{\mathscr{H}_1} = \int_{\mathscr{H}_1} \mathrm{H}^\mu U_\mu \geq -\nu(Q_m, Q_0, \ell_0, \ell, \rho_0) \mathcal{A}_{\mathscr{H}_1}$$

The smeared null energy condition

Null QEIs

Can we prove a QEI over a null geodesic?

$$\int d\lambda \langle :T_{\mu\nu}: U^\mu U^\nu \rangle_\omega f^2(\lambda) \geq -A \int d\lambda f'(\lambda)^2$$

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Considered a sequence of vacuum-plus-two-particle states in which the three-momenta of excited modes are unbounded and become more and more parallel to the spatial part of the null vector U^μ. [Fewster, Roman, 2002]

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Idea

In quantum field theory there is often an ultraviolet cutoff ℓ_{UV} which restricts the three-momenta. We can write $G_N \lessapprox \ell_{\mathrm{UV}}^2/N$.

The smeared null energy condition

Smeared null energy condition (SNEC) for the minimally coupled scalar field in four dimensional Minkowski spacetime [Freivogel, Krommydas, 2018]

$$\int d\lambda \langle :T_{\mu\nu}: U^\mu U^\nu \rangle_\omega f^2(\lambda) \geq -\frac{4B}{G_N} \|f'\|^2$$

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- $\ell_{\mathrm{UV}} \approx$ Planck length $\to$ B order 1 $\to$ A lot of negative energy allowed
- $\ell_{\mathrm{UV}} \gg$ Planck length $\to$ B small $\to$ A little negative energy allowed

Null semiclassical singularity theorem

Semiclassical Einstein equation

$$8\pi G_N \langle :T_{\mu\nu}: U^\mu U^\nu \rangle_\omega = R_{\mu\nu} U^\mu U^\nu$$

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[Freivogel, E-AK, Krommydas, 2020]

(i) Energy condition

$$\int d\lambda f^2(\lambda) R_{\mu\nu} U^\mu U^\nu \geq -Q_1 ||f'||^2, \quad Q_1 = 32\pi B$$

and $R_{\mu\nu}\ell^\mu\ell^\nu \leq 0$ holds for $\lambda \in [-\ell_0, 0]$

(ii) The mean normal curvature of P satisfies

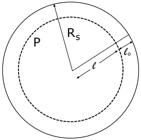
$$H \leq -\nu(B, \ell_0, \ell)$$

(iii) There exists a non-compact Cauchy surface.

$\Rightarrow$ The spacetime is null geodesically incomplete.

Estimation of the mean normal curvature

Toy model of evaporating black holes



Affine distance $\rightarrow$ Coordinate distance

- $\ell \rightarrow yR_s$
- $\ell_0 \rightarrow xR_s$

Strategy: compare H of Schwarzschild geometry to $\nu(B, \ell_0, \ell)$ from theorem

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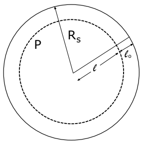

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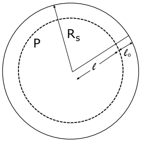

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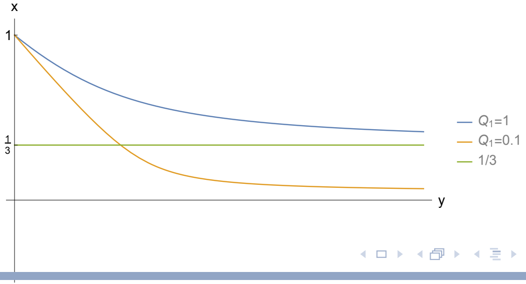

Null semiclassical area theorem

Theorem [E-AK, Sacchi, 2023]

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$$\int d\lambda f^2(\lambda) R_{\mu\nu} U^\mu U^\nu \geq -Q_1 ||f'||^2$$

and $R_{\mu\nu} U^\mu U^\nu \geq \rho_0$ holds for $\lambda \in [0, \ell_0]$

(ii) Causality condition: M is strongly asymptotically predictable

Then

$$\delta_U \mathcal{A}_{\mathscr{H}_1} = \int_{\mathscr{H}_1} \mathrm{H}^\mu U_\mu \geq -\nu(B, \ell_0, \ell, \rho_0) \mathcal{A}_{\mathscr{H}_1}$$

Estimation of evaporation rate

Evaporation rate for spherical black holes ($\alpha \sim 2 \times 10^{-4}$)

$$\nu_{\text{ev}} = -\frac{1}{M}\frac{dM}{dt} = (8\pi)^3 \alpha \left(\frac{k}{T_{\text{pl}}^2 \hbar} \right) T^3 \,.$$

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we have from the theorem for spherical black holes

$$-\frac{\delta_U \mathcal{A}_{\mathscr{H}}}{\mathcal{A}_{\mathscr{H}}} = -\frac{1}{M}\frac{dM}{dt} \leq \nu_{\mathrm{opt}}(B, T) = \frac{2\sqrt{2\pi^3}}{\sqrt{5}} \sqrt{B} \left(\frac{k}{\hbar T_{\mathrm{pl}}}\right) T^2 .$$

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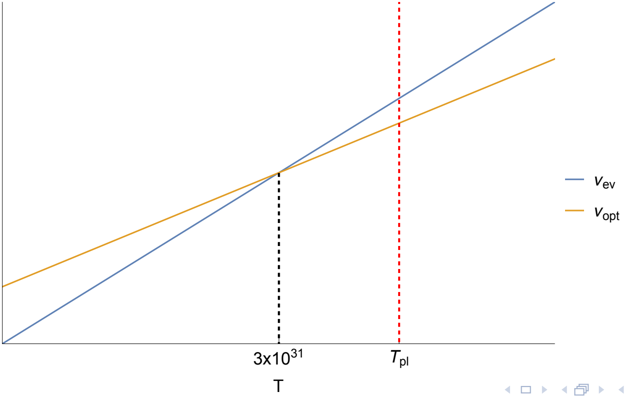
$$\nu_{\rm ev} = -\frac{1}{M}\frac{dM}{dt} = (8\pi)^3\alpha\left(\frac{k}{T_{\rm pl}^2\hbar}\right) T^3\,, \quad \nu_{\rm opt}(B,T) = \frac{2\sqrt{2\pi^3}}{\sqrt{5}}\sqrt{B}\left(\frac{k}{\hbar T_{\rm pl}}\right) T^2$$



Conclusions

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Better null energy conditions? DSNEC

$$\int d^2x^{\pm} f^2(x^{\pm})\langle :T_{--}:\rangle_{\omega} \geq -\frac{\mathcal{N}}{(\delta^{+})^{n/2-1}(\delta^{-})^{n/2+1}}$$

[Fliss, Freivogel, E-AK, 2021]