Tomimatsu vs Smarr: A Tale of the Masses

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Protagonists

- The usefulness of Tomimatsu's integral mass formula for computing the masses of systems with black holes, including the Kerr-Newman solution, is widely recognized.
- On the other hand, the Smarr mass relation has proven right on a plethora of black hole solutions.

However, it has been argued that Tomimatsu's integral mass formula fails to reproduce the known Smarr relation for dyonic Kerr-Newman black holes. The existence of contradictory results between two well-established concepts is intriguing, which is the focus of our investigation.

Tomimatsu's integral mass formula

We introduce Tomimatsu's integral mass formula[2]

$$M = -\frac{1}{8\pi}\int_H \omega\partial_z\chi \, d\varphi dz, \tag{1}$$

which we'll be explain in detail by introducing the Ernst formalism. Keep an eye for ω and χ.

[2] Tomimatsu, "Equilibrium of two rotating charged black holes and the Dirac string".

Papapetrou line element

The main idea of Ernst's formalism is to use the Papapetrou line element,

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{2}$$

with the coordinate system $\{t, \rho, \varphi, z\}$, which describes a generic stationary axisymmetric electrovac solution, its three unknown functions f, γ and ω depending only on ρ and z.

Ernst's equations

The Einstein-Maxwell equations can be reduced to a system of two differential equations for the complex potentials $\mathcal{E}$ and ϕ[3][4]:

$$
\begin{aligned}
(\mathrm{Re}\mathcal{E} + \phi\overline{\phi})\Delta\mathcal{E} &= (\nabla\mathcal{E} + 2\overline{\phi}\nabla\phi)\nabla\mathcal{E}, \\
(\mathrm{Re}\mathcal{E} + \phi\overline{\phi})\Delta\phi &= (\nabla\mathcal{E} + 2\overline{\phi}\nabla\phi)\nabla\phi,
\end{aligned} \tag{3}
$$

where Δ and ∇ are the usual three-dimensional Laplacian and gradient operators, respectively, and a bar over a symbol means complex conjugation.

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[3] Ernst, "New formulation of the axially symmetric gravitational field problem".

[4] Ernst, "New formulation of the axially symmetric gravitational field problem. II".

The complex potentials ϕ and $\mathcal{E}$

The potentials $\mathcal{E}$ and ϕ are related to the metric functions f, ω and to the φ and t components of the electromagnetic four-potential $A = A_t dt + A_\varphi d\varphi$ by the equations

$$\mathcal{E} = f - \phi\bar{\phi} + i\chi, \quad \phi = -A_t + iA'_\varphi, \tag{4}$$

and by the systems of the first-order differential equations

$$\begin{aligned} \partial_\rho \omega &= -\rho f^{-2}[\partial_z \chi + 2\mathrm{Im}(\bar{\phi}\partial_z \phi)], \\ \partial_z \omega &= \rho f^{-2}[\partial_\rho \chi + 2\mathrm{Im}(\bar{\phi}\partial_\rho \phi)], \end{aligned} \tag{5}$$

and

$$\partial_\rho A_\varphi^{'} = \rho^{-1} f(\partial_z A_\varphi + \omega \partial_z A_t),$$
$$\partial_z A_\varphi^{'} = -\rho^{-1} f(\partial_\rho A_\varphi + \omega \partial_\rho A_t), \tag{6}$$

so that the knowledge of $\mathcal{E}$ and Φ permits one to find the functions f, ω, A_t and A_φ, while for the determination of the remaining metric function γ one has to solve the system

$$\begin{aligned}
\partial_\rho \gamma = {} & \frac{1}{4}\rho f^{-2}[(\partial_\rho \mathcal{E} + 2\bar{\Phi}\partial_\rho \Phi)(\partial_\rho \bar{\mathcal{E}} + 2\Phi\partial_\rho \bar{\Phi}) \\
& - (\partial_z \mathcal{E} + 2\bar{\Phi}\partial_z \Phi)(\partial_z \bar{\mathcal{E}} + 2\Phi\partial_z \bar{\Phi})] \\
& - \rho f^{-1}(\partial_\rho \Phi \partial_\rho \bar{\Phi} - \partial_z \Phi \partial_z \bar{\Phi}), \\
\partial_z \gamma = {} & \frac{1}{2}\rho f^{-2}\mathrm{Re}[(\partial_\rho \mathcal{E} + 2\bar{\Phi}\partial_\rho \Phi)(\partial_z \bar{\mathcal{E}} + 2\Phi\partial_z \bar{\Phi})] \\
& - 2\rho f^{-1}\mathrm{Re}(\partial_\rho \bar{\Phi}\partial_z \Phi),
\end{aligned} \tag{7}$$

Smarr relation

We introduce the well-known and well-established Smarr relation,

$$m = \frac{\kappa}{8\pi} A + 2J\Omega^H + q\Phi_e^H + p\Phi_m^H, \tag{8}$$

where all the quantities have their usual interpretation.

The energy-momentum tensor

However, the validity of the mass formula (1) in the presence of magnetic charge was questioned in the paper[5], where the authors used during their calculations the conventional representation of the electromagnetic energy-momentum tensor

$$T^{\mu}{}_{\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta}\right), \tag{9}$$

which led them to a specific dyonic configuration with two magnetic Dirac strings and an additional electromagnetic term in the integrand of (1), both strings carrying portions of nonzero mass, so that the mass parameter m becomes the sum of three different contributions – one coming from the surface integral evaluated on the horizon, and two others arising from the singular "massive" Dirac strings.

[5] Clément and Gal'tsov, "On the Smarr formula for rotating dyonic black holes".

Revisiting Tomimatsu

Returning to Tomimatsu's mass formula

$$M = -\frac{1}{8\pi}\int_H \omega \partial_z \chi \, d\varphi dz, \tag{10}$$

we can observe that the contributions are entirely from the black hole's horizon, without taking into account any contributions from the Dirac's strings.

Komar mass

In Tomimatsu's paper[6], Tomimatsu started with the same standard integral for the calculation of the Komar mass that has been recently used in the papers[7][8]:

$$M_K = \frac{1}{4\pi}\int_{\infty} D^{\nu}k^{\mu}d\Sigma_{\mu\nu} = \frac{1}{4\pi}\int_{\partial\mathcal{M}} D^{\nu}k^{\mu}d\Sigma_{\mu\nu} + \frac{1}{4\pi}\int_{\mathcal{M}} D_{\nu}D^{\nu}k^{\mu}dS_{\mu}, \tag{11}$$

with the same decomposition into the surface and bulk integrals.

[6] Tomimatsu, "Equilibrium of two rotating charged black holes and the Dirac string".

[7] Ramírez-Valdez, García-Compeán, and Manko, "Dyonic black holes in the theory of two electromagnetic potentials. I".

[8] Clément and Gal'tsov, "On the Smarr formula for rotating dyonic black holes".

By choosing the horizon of the black hole as $\partial\mathcal{M}$, Tomimatsu computed the first integral on the right-hand side of (11) and obtained

$$\begin{aligned}\frac{1}{4\pi}\int_H D^\nu k^\mu d\Sigma_{\mu\nu} &= \frac{1}{8\pi}\int_H[-\omega\partial_z\chi + 2\omega\mathrm{Im}(\Phi\partial_z\bar{\Phi})]d\varphi dz\\ &= \frac{1}{8\pi}\int_H[-\omega\partial_z\chi + 2\omega(A_t\partial_z\tilde{A}_t - \tilde{A}_t\partial_z A_t)]d\varphi dz, \qquad (12)\end{aligned}$$

and he also rewrote the bulk integral on the right-hand side of (11) in the form

$$\frac{1}{4\pi}\int_{\mathcal{M}} D_\nu D^\nu k^\mu dS_\mu = -2\int_{\mathcal{M}} T^t{}_t\sqrt{-g}\,d^3x, \qquad (13)$$

and the correctness of these formulas was not objected in[9].

[9] Clément and Gal'tsov, "On the Smarr formula for rotating dyonic black holes".

The bulk term

The authors of[10], however, questioned Tomimatsu's result of computing the integral on the right-hand side of (13), namely,

$$-2\int_{\mathcal{M}} T^{t}{}_{t}\sqrt{-g}\, d^3x = -\frac{1}{4\pi}\int_{H} \omega \mathrm{Im}(\Phi\partial_z\bar{\Phi})d\varphi dz, \tag{14}$$

which, together with (12), gives formula (1). Though they rightly pointed out that the representation (9) of the energy-momentum tensor used by Tomimatsu requires additionally taking account of two singular string sources, which modifies the horizon contribution (1) of the Komar mass.

[10] Clément and Gal'tsov, "On the Smarr formula for rotating dyonic black holes".

Remarkably, the validity of Tomimatsu's mass integral (1) can be readily demonstrated by employing the symmetrical representation of the electromagnetic energy-momentum tensor

$$T^{\mu}{}_{\nu} = \frac{1}{8\pi}(F^{\mu\alpha}F_{\nu\alpha} + \tilde{F}^{\mu\alpha}\tilde{F}_{\nu\alpha}), \tag{15}$$

for the evaluation of the integral on the right-hand side of (13). Then, following the steps outlined in the paper[11] for that representation, the bulk integral on the left-hand side of (14) reduces to the surface integral over the horizon, yielding

$$-2\int_{\mathcal{M}} T^{t}{}_{t}\sqrt{-g}\,d^3x = \frac{1}{4\pi}\int_{H}(A_t\partial_z\tilde{A}_\varphi - \tilde{A}_t\partial_z A_\varphi)d\varphi dz, \tag{16}$$

where, at the last stage of the computation, we have used the substitutions

$$\begin{aligned} \rho^{-1}f[(\rho^2f^{-2}-\omega^2)\partial_\rho A_t - \omega\partial_\rho A_\varphi] &= -\partial_z\tilde{A}_\varphi \\ \rho^{-1}f[(\rho^2f^{-2}-\omega^2)\partial_\rho \tilde{A}_t - \omega\partial_\rho \tilde{A}_\varphi] &= \partial_z A_\varphi. \end{aligned} \tag{17}$$

[11] Ramírez-Valdez, García-Compeán, and Manko, "Dyonic black holes in the theory of two electromagnetic potentials. I".

Now, combining formulas (12) and (16) in one, and also taking into account that ω assumes a constant value on the horizon, we get for the Komar integral (11) the expression

$$M_K = \frac{1}{8\pi}\int_H [-\omega\partial_z\chi + 2\omega A_t\partial_z(\tilde{A}_t + \omega^{-1}\tilde{A}_\varphi) + 2\omega\tilde{A}_t\partial_z(-A_t - \omega^{-1}A_\varphi)]d\varphi dz, \quad (18)$$

and lastly, after noting that the second and third terms in the integrand of (18) vanish because these contain the derivatives of the potentials Φ_e and Φ_m, both potentials taking constant values on the horizon, we obtain the final expression for the Komar mass

$$M_K = -\frac{1}{8\pi}\int_H \omega\partial_z\chi\, d\varphi dz, \quad (19)$$

which fully coincides with Tomimatsu's formula (1).

Conclusions

We have shown that the use of a specific representation of the electromagnetic energy-momentum tensor is able to provoke erroneous interpretations of the physical properties of dyonic black holes: thus, the choice of the canonical representation (9) for $T^{\mu}{}_{\nu}$ in the Komar mass integral leads to the appearance of magnetic Dirac strings as the sources of mass, while the representation of $T^{\mu}{}_{\nu}$ involving only the dual electromagnetic tensor $\tilde{F}^{\mu\nu}$[12] gives rise to massive Dirac strings generated by the electric charge. This naturally singles out the symmetrical representation (15) of $T^{\mu}{}_{\nu}$ as the most appropriate one for the dyonic solutions because no contributions due to string singularities emerge during the evaluation of the mass integral.

[12] Ramírez-Valdez, García-Compeán, and Manko, "Dyonic black holes in the theory of two electromagnetic potentials. I".

Conclusions

It follows directly from our analysis that Dirac strings (magnetic and electric ones) must be excluded from the physical picture of dyonic spacetimes. Nevertheless, the semi-infinite singularities that are characteristic mathematical attributes of the components A_φ and $\tilde{A}_\varphi$ in the presence of nonzero magnetic and electric net charges still remain a legitimate part of the general mathematical toolkit and are expected to be taken into account as purely mathematical objects in some calculations involving the functions A_φ and $\tilde{A}_\varphi$.

Tomimatsu=Smarr

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