Introduction to Modified Gravity

1. Motivation
2. GR and Lovelock's theorem
3. How to go beyond GR

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2. GR and Lovelock’s theorem
3. How to go beyond GR

MOTIVATION

Why modified gravity?

A motivation for IR modification

- Gravity at long distances

  Flattening galaxy rotation curves

  extra gravity

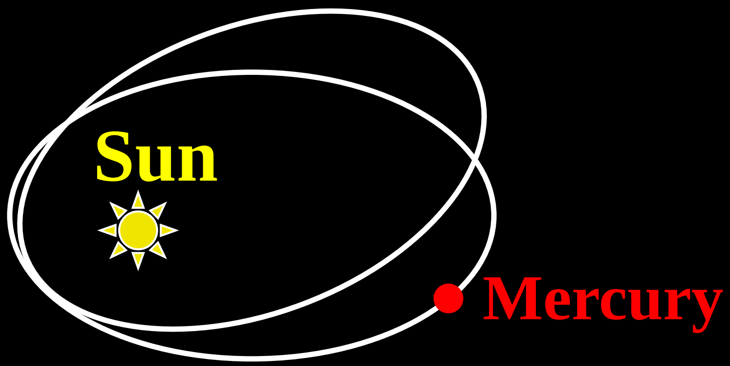
  Dimming supernovae

  accelerating universe

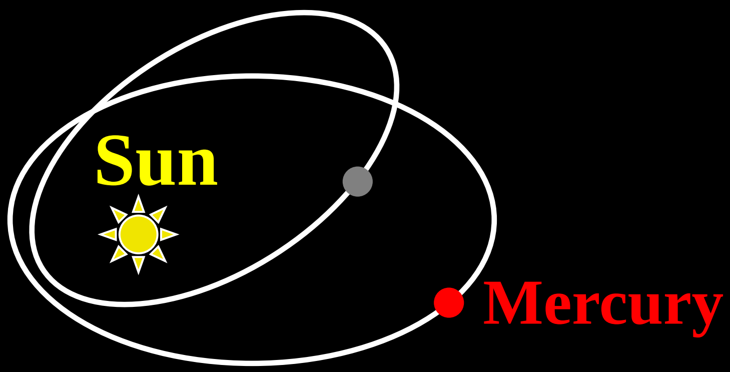
- Usual explanation: new forms of matter (DARK MATTER) and energy (DARK ENERGY).

Dark component in the solar system?

Precession of perihelion observed in 1800's…



which people tried to explain with a "dark planet", Vulcan,

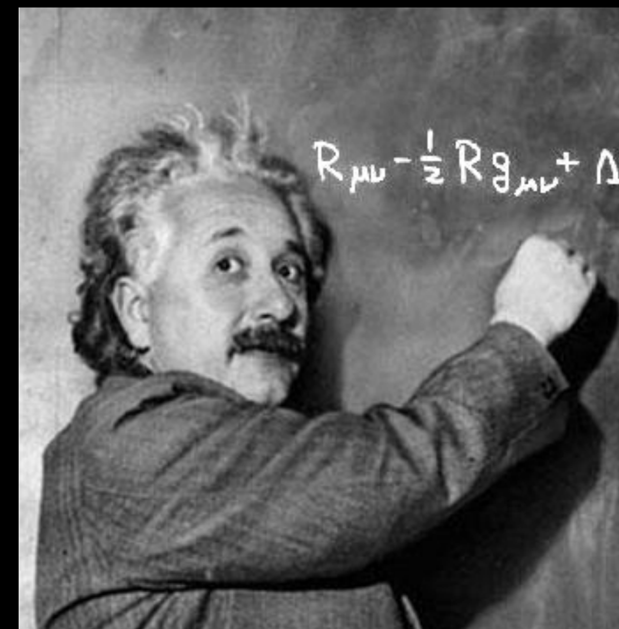


But the right answer wasn't "dark planet", it was "change gravity" from Newton to GR.

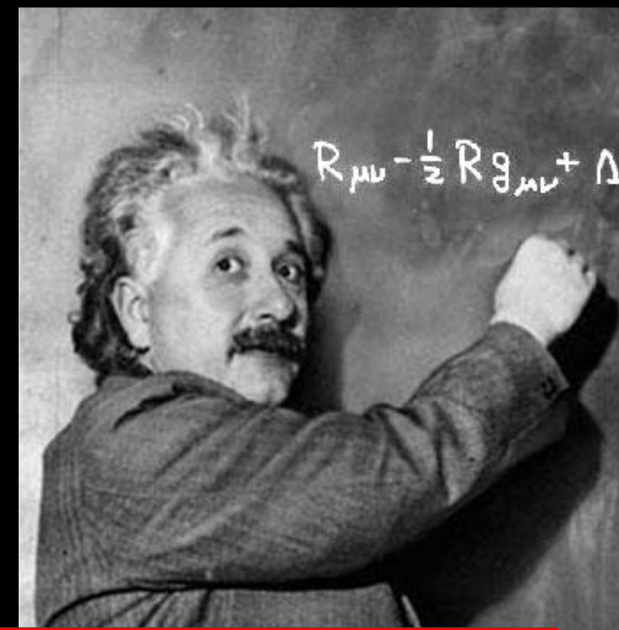
Why modified gravity?

- Can we address **mysteries in the universe**?
  Dark energy, dark matter, inflation, big-bang singularity, cosmic magnetic field, etc.

How to unify Quantum Theory with General Relativity?



How to unify Quantum Theory with General Relativity?



Probably we need to modify GR at short distances

Why modified gravity?

- Can we address **mysteries in the universe**?
  Dark energy, dark matter, inflation, big-bang singularity, cosmic magnetic field, etc.
- Help constructing a **theory of quantum gravity**?
  Superstring, Horava-Lifshitz, etc.
- Do we really **understand GR**?
  One of the best ways to understand something may be to break (modify) it and then to reconstruct it.
- …

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GENERAL RELATIVITY AND LOVELOCK’S THEOREM

Equivalence principle and metric theories of gravity

- **Weak equivalence principle (WEP)**

uncharged test body
initial event in spacetime
initial velocity

$\Rightarrow$ subsequent trajectory is independent of internal structure & composition

- **Einstein's equivalence principle (EEP)**

i) **WEP is valid**
ii) outcome of any local nongravitational test experiment is
independent of the velocity of freely falling apparatus
and of the time and position in the universe
Basically saying that **gravity $\simeq$ acceleration**

- **EEP → metric theory**

EEP → validity of special relativity in local free-falling frame
→ **$\exists$ tensor $g_{\mu\nu}$ that reduces to $\eta_{\mu\nu}$ in local free-falling frame**
This argument does not exclude existence of other metrics.

Einstein's theory

- **Assumptions**
  EEP (→ ∃metric $g_{\mu\upsilon}$)
  gravity is described by the metric **$g_{\mu\upsilon}$ only**
- **Invariant action** $\boldsymbol{I = \int d^4x \sqrt{-g}\, L}$
  L : scalar made of **$g_{\mu\upsilon}$ & its derivatives**
  up to 1st derivatives → constant only
  up to 2nd derivatives → scalar made of $g_{\mu\upsilon}$ & $R_{\mu\upsilon\rho\sigma}$
  …
- **Ingredients in L**
  $1, R, R^2, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \nabla^\mu R \nabla_\mu R, R^3, \cdots$

  ⬇ scale M

  $$L = c_0 M^4 + c_1 M^2 R + c_2 R^2 + c_3 R^{\mu\nu}R_{\mu\nu} + c_4 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + \cdots$$

  ⬇ **truncate @ terms with two derivatives**

  $$\boldsymbol{L = c_0 M^4 + c_1 M^2 R = \frac{M_{Pl}^2}{2}(R - 2\Lambda)} \qquad \left(M_{Pl}^2 = 2c_1 M^2, \Lambda = -\frac{c_0}{2c_1}M^2\right)$$

  This is Einstein-Hilbert action!

  c.f. cosmological constant problem = "Why $\left|\frac{c_0}{4c_1^2}\right| \ll 1$?"

Einstein's theory

- **Field equation**

$$I_{EH} = \frac{\mathrm{M_{Pl}^2}}{2}\int d^4x\,\sqrt{-g}(\mathrm{R} - 2\Lambda)$$

$$\delta\left(\sqrt{-g}\right) = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \quad (\leftarrow \delta\,(\ln\det A) = \delta(Tr\ln A) = Tr(A^{-1}\delta A))$$

$$\delta\left(\sqrt{-g}R\right) = \sqrt{-g}\{-G^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\left[\nabla^\nu\delta g_{\mu\nu} - \nabla_\mu\left(g^{\rho\sigma}\delta g_{\rho\sigma}\right)\right]\}$$

$$\therefore\ \delta I_{EH} = \frac{\mathrm{M_{Pl}^2}}{2}\int d^4x\,\sqrt{-g}\left[-(G^{\mu\nu} + \Lambda g^{\mu\nu})\delta g_{\mu\nu}\right]$$

$$\boldsymbol{I_{tot} = I_{EH} + I_{matter}}$$

$$\delta I_{matter} = \int d^4x\,\left[\frac{\sqrt{-g}}{2}T^{\mu\nu}\delta g_{\mu\nu} + (matter\ eom)\delta(matter)\right]$$

$$(T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta I_{matter}}{\delta g_{\mu\nu}})$$

$\boldsymbol{\delta I_{tot} = 0}$ ➡ $\boldsymbol{M_{Pl}^2(G^{\mu\nu} + \Lambda g^{\mu\nu}) = T^{\mu\nu}}$

Einstein eq with $\boldsymbol{G_N = \frac{1}{8\pi M_{Pl}^2}}$

Uniqueness of GR (ref. D.Lovelock, Aquationes Mathematica 4 (1970) 127)

- Lovelock's theorem

(i) $A^{\mu\upsilon}$ is a symmetric tensor (μ,ν = 0,1,2,3)
(ii) $A^{\mu\upsilon} = A^{\mu\upsilon}(g_{\rho\sigma}, g_{\rho\sigma,\alpha}, g_{\rho\sigma,\alpha\beta})$ (++++) (valid for (-+++) as well)
(iii) $A^{\mu\upsilon}{}_{;\nu} = 0$ (;ν represents covariant derivative)
(iv) 4-dimensions

➡ $A^{\mu\nu} = a\, G^{\mu\nu} + b g^{\mu\nu}$ (a, b: constants, $G^{\mu\nu}$: Einstein tensor)

- Motivation for assumptions (i)-(iii)

(i) $A^{\mu\nu}$ is to be EOM for $g_{\mu\upsilon}$ and thus should be symmetric.
(ii) If EOM depends on 3rd or higher derivatives of $g_{\mu\upsilon}$ then # of d.o.f. (in Lorentzian case) may increase.
(iii) If $^\exists I$ s.t. $A^{\mu\nu} = \frac{1}{\sqrt{|g|}}\frac{\delta I}{\delta g_{\mu\nu}}$ and if I is diffeo invariant then $A^{\mu\nu}{}_{;\nu} = 0$.

$$\left(\begin{array}{l} \because\ g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu},\ \delta g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} \\ 0 = \delta I = \int d^4x \frac{\delta I}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = -2\int d^4x \sqrt{g}\, \xi_\nu A^{\mu\nu}{}_{;\nu} \quad \text{for} {}^\forall \xi_\nu \end{array} \right)$$

c.f. "symmetric" in (i) can be dropped (J.Math.Phys. 13, 874 (1972)) .

Uniqueness of GR (ref. D.Lovelock, Aquationes Mathematica 4 (1970) 127)

- What Lovelock actually proved

In n-dim. Lovelock proved theorems 1 & 2 below

- **Theorem 1**

(i)-(iii) ➡ $\boldsymbol{A^{\mu\nu} = \sum_{k=1}^{m-1} c_k \theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}} \prod_{h=1}^{k} R_{\alpha_{4h-1}\alpha_{4h-3}\alpha_{4h-2}\alpha_{4h}} + b g^{\mu\nu}}$

$$\mathrm{m} = \begin{cases} \frac{n}{2} & (n: even) \\ \frac{1}{2}(n+1) & (n: odd) \end{cases}, \qquad c_k, b : \text{const.}$$

$\theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}}$ $(\mathrm{k} = 1, \cdots, m-1)$: a tensor satisfying (a)-(d) below

(a) $\theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}} = \theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}}(g_{\rho\sigma})$

(b) symmetric in (ij) and in $(i_{2h-1}i_{2h})$ for $h = 1, \cdots, 2k$

(c) symmetric under interchange of the pair $(\mu\nu)$ with the pair $(\alpha_{2h-1}\alpha_{2h})$ for all $h = 1, \cdots, 2k$

(d) the cyclic sum involving any three of the four indices $(\mu\nu)$ $(\alpha_{2h-1}\alpha_{2h})$ for $h = 1, \cdots, 2k$ vanishes

(c) follows from (b)&(d)

$$0 = \theta^{\mu(\nu\alpha_1\alpha_2)} + \theta^{\nu(\alpha_1\alpha_2\mu)} - \theta^{\alpha_1(\alpha_2\mu\nu)} - \theta^{\alpha_2(\mu\nu\alpha_1)} = \frac{1}{3}(\theta^{\mu\nu\alpha_1\alpha_2} - \theta^{\alpha_1\alpha_2\mu\nu})$$

Uniqueness of GR (ref. D.Lovelock, Aquationes Mathematica 4 (1970) 127)

- **Theorem 2**

p : positive integer

$\psi^{\mu\nu\alpha_1\cdots\alpha_{2p}}$ is a tensor with the following properties (a)'-(d)'

(a)' : (a) with $\theta \to \psi, 4k \to 2p$

(b)'-(d)' : (b)-(d) with $2k \to p$

➡ $(n-p)\boxed{\psi^{\mu\nu\alpha_1\cdots\alpha_{2p}}} = g^{\mu\nu}\boxed{g_{\rho\sigma}\psi^{\rho\sigma\alpha_1\cdots\alpha_{2p}}} - \frac{1}{2}\sum_{h=1}^{2p} g^{\alpha_h\nu}\boxed{g_{\rho\sigma}\psi^{\rho\sigma\alpha_1\cdots\alpha_{h-1}\mu\alpha_{h+1}\cdots\alpha_{2p}}}$

Theorem 2 shows a way to calculate $\theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}}$ defined in Theorem 1 and its uniqueness (up to an overall constant factor).

- **Corollary 1**

n=2 → $A^{\mu\nu} = bg^{\mu\nu} \quad (b\colon const.)$

(proof of corollary 1)

m=1 for n=2 ∴ corollary 1 follows from theorem 1 Q.E.D.

- **Corollary 2**

n=3 or 4 → $A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu} \quad (a, b\colon const.)$

Corollary 2 for n=4 is what is usually known as Lovelock's theorem.

Uniqueness of GR (ref. D.Lovelock, Aquationes Mathematica 4 (1970) 127)

(proof of corollary 2)

- m=2 for n=3 or 4

$$\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4} \equiv g_{\rho\sigma}\theta^{\rho\sigma\alpha_1\alpha_2\alpha_3\alpha_4}$$

- Theorem 2 with p=2

$$(n-2)\theta^{\mu\nu\alpha_1\alpha_2\alpha_3\alpha_4} = g^{\mu\nu}\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4} - \frac{1}{2}\Big(g^{\alpha_1\nu}\tilde{\theta}^{\mu\alpha_2\alpha_3\alpha_4} + g^{\alpha_2\nu}\tilde{\theta}^{\alpha_1\mu\alpha_3\alpha_4} + g^{\alpha_3\nu}\tilde{\theta}^{\alpha_1\alpha_2\mu\alpha_4} + g^{\alpha_4\nu}\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\mu}\Big)$$

with p=1

$$(n-1)\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4} = g^{\alpha_1\alpha_2}\tilde{\tilde{\theta}}^{\alpha_3\alpha_4} - \frac{1}{2}\left(g^{\alpha_3\alpha_2}\tilde{\tilde{\theta}}^{\alpha_1\alpha_4} + g^{\alpha_4\alpha_2}\tilde{\tilde{\theta}}^{\alpha_3\alpha_1}\right)$$

$$\tilde{\tilde{\theta}}^{\alpha_1\alpha_2} \equiv g_{\rho\sigma}\tilde{\theta}^{\rho\sigma\alpha_1\alpha_2}$$

$\tilde{\tilde{\theta}}^{\alpha_1\alpha_2}(g_{\mu\nu})$ is a symmetric tensor → $\tilde{\tilde{\theta}}^{\alpha_1\alpha_2} = \tilde{a}g^{\alpha_1\alpha_2}$ $(\tilde{a} : const.)$

[lemma A2 of D.Lovelock, Arch.Ratl.Mech.Anal. 33 (1969) 54 restricted to symmetric part]

➡ $$2(n-2)\theta^{\mu\nu\alpha_1\alpha_2\alpha_3\alpha_4}R_{\alpha_3\alpha_1\alpha_2\alpha_4} = 2g^{\mu\nu}\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4}R_{\alpha_3\alpha_1\alpha_2\alpha_4} - \tilde{\theta}^{\mu\alpha_2\alpha_3\alpha_4}R_{\alpha_3}{}^{\nu}{}_{\alpha_2\alpha_4} - \tilde{\theta}^{\alpha_1\mu\alpha_3\alpha_4}R_{\alpha_3\alpha_1}{}^{\nu}{}_{\alpha_4} - \tilde{\theta}^{\alpha_1\alpha_2\mu\alpha_4}R^{\nu}{}_{\alpha_1\alpha_2\alpha_4} - \tilde{\theta}^{\alpha_1\alpha_2\alpha_3\mu}R_{\alpha_3\alpha_1\alpha_2}{}^{\nu}$$

$$\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4}R_{\alpha_3\alpha_1\alpha_2\alpha_4} = -\frac{3\tilde{a}}{2(n-1)}R$$

$$\tilde{\theta}^{\mu\alpha_2\alpha_3\alpha_4}R_{\alpha_3}{}^{\nu}{}_{\alpha_2\alpha_4} = \tilde{\theta}^{\alpha_1\mu\alpha_3\alpha_4}R_{\alpha_3\alpha_1}{}^{\nu}{}_{\alpha_4} = \tilde{\theta}^{\alpha_1\alpha_2\mu\alpha_4}R^{\nu}{}_{\alpha_1\alpha_2\alpha_4} = \tilde{\theta}^{\alpha_1\alpha_2\alpha_3\mu}R_{\alpha_3\alpha_1\alpha_2}{}^{\nu} = -\frac{3\tilde{a}}{2(n-1)}R^{\mu\nu}$$

➡ $$\theta^{\mu\nu\alpha_1\alpha_2\alpha_3\alpha_4}R_{\alpha_3\alpha_1\alpha_2\alpha_4} = \frac{3\tilde{a}}{(n-1)(n-2)}G^{\mu\nu}$$

- Theorem 1

➡ $$A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu} \qquad a = \frac{3c_1\tilde{a}}{(n-1)(n-2)}$$

<u>Q.E.D.</u>

HOW TO GO BEYOND GR

How to go beyond GR?

- Lovelock's theorem (to be more precise, corollary 2) assumes
  - 4-dim. (pseudo-)Riemannian geometry
  - the metric is the only physical field

(The theorem is at the level of eoms.)

- Modification of GR (at the level of eoms) then requires at least one of the following
  - extra dimension
  - extra dof.
  - Lorentz violation
  - non (pseudo-)Riemannian geometry

There are many possibilities to explore!

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- Can we address **mysteries in the universe**?
  Dark energy, dark matter, inflation, big-bang singularity, cosmic magnetic field, etc.
- Help constructing a **theory of quantum gravity**?
  Superstring, Horava-Lifshitz, etc.
- Do we really **understand GR**?
  One of the best ways to understand something may be to break (modify) it and then to reconstruct it.
- …

Some examples (my personal experiences)

I. Effective field theory (EFT) approach
IR modification of gravity
motivation: dark energy/inflation, universality

II. Massive gravity
IR modification of gravity
motivation: “Can graviton have mass?” & dark energy

III. Minimally modified gravity
IR modification of gravity
motivation: tensions in cosmology, various constraints

IV. Horava-Lifshitz gravity
UV modification of gravity
motivation: quantum gravity

V. Superstring theory
UV modification of gravity
motivation: quantum gravity, unified theory