Lionel ***London***

Review of non-Hermitian aspects of black hole quasi-normal modes I

UK Royal Society

KING'S College LONDON

A brief historical overview

- Practical connections between black hole physics and e.g. quantum mechanics have long been observed
- Single perturbed black holes are a key setting for non-Hermitian physics

1915 — 1970s
Einstein
Vish., Teukolsky+

1 Spacetime is dissipative. Black holes leak energy when perturbed via "Quasi-normal modes" (QNMs)

1970s — 1990s
Leaver, Leung, +

2 First analytic descriptions of QNMs and related scalar products

1990s — 2005
Baker, Pretorious+

3 First simulations of black hole mergers in Einstein's GR

2015 — 2025
LIGO-Virgo-Kagra

4 Routine detection of Gravitational Waves (GWs)

mid-2030s & Beyond
LISA, Einstein Telescope, Cosmic Explorer,+

5 High precision GW Spectroscopy with QNMs — likely first conclusive deviations from GR

A brief historical overview

- Today the GW community is preparing tools for the **2030s**
- Aim: to conclusively observe physics **beyond Einstein's GR**
- For this, the non-Hermitian nature of single perturbed black holes is likely to play a key role (e.g. BH Spectroscopy)

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Teukolsky & Press, +

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From Einstein to the first gravitational wave detection

Roy Kerr 1963
Spinning or "Kerr" black hole

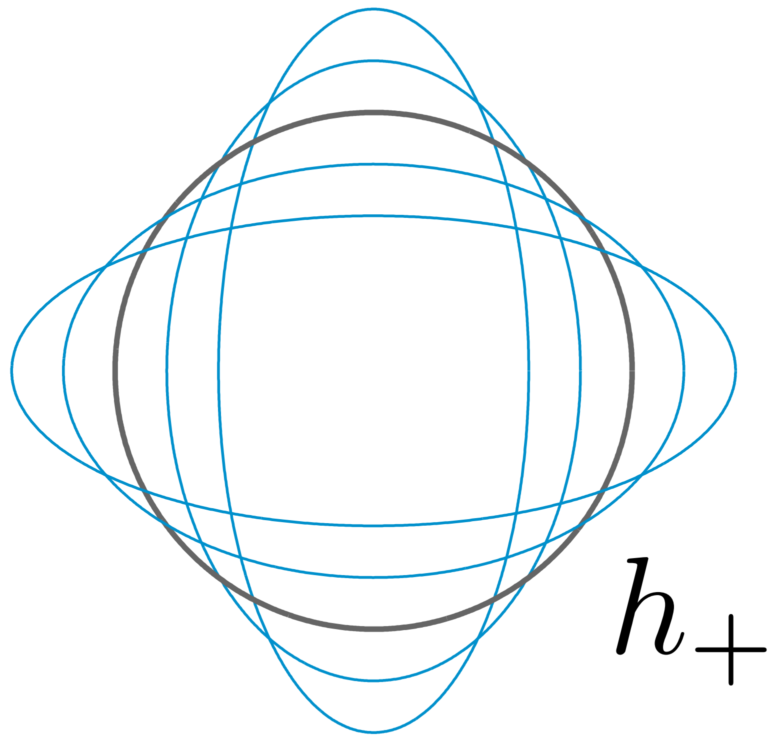
Newman & Penrose 1966
Spin Weighted Harmonics

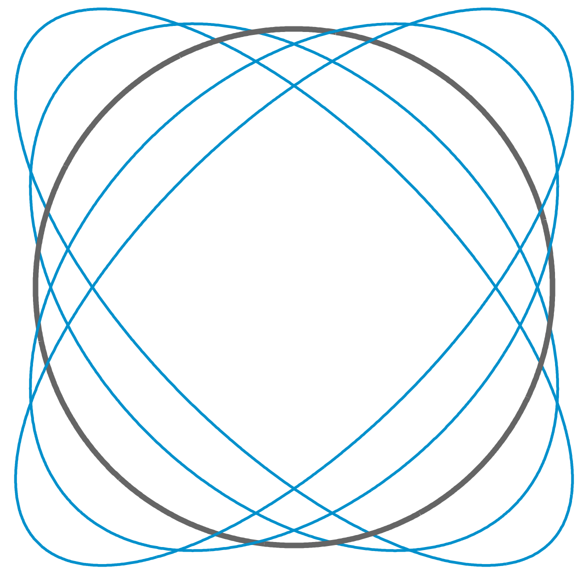
- **Newman & Penrose** where interested in finding mathematically **robust and natural ways of representing spacetime structure**
- They are credited with introducing the **"spin-weighted" spherical harmonics** to GW theory
- **Application**: h_+ and $h_\times$ are **physical observables** encoded in $h_{\mu\nu}$

$$h(t,r,\theta,\phi) = h_+ + ih_\times = \frac{1}{r}\sum_{\ell,|m|\leq\ell} h_{\ell m}(t)\ {}_{-2}Y_{\ell m}(\theta)e^{i\phi m}$$

Spherical Harmonic: ${}_{-2}Y_{\ell m}(\theta)$

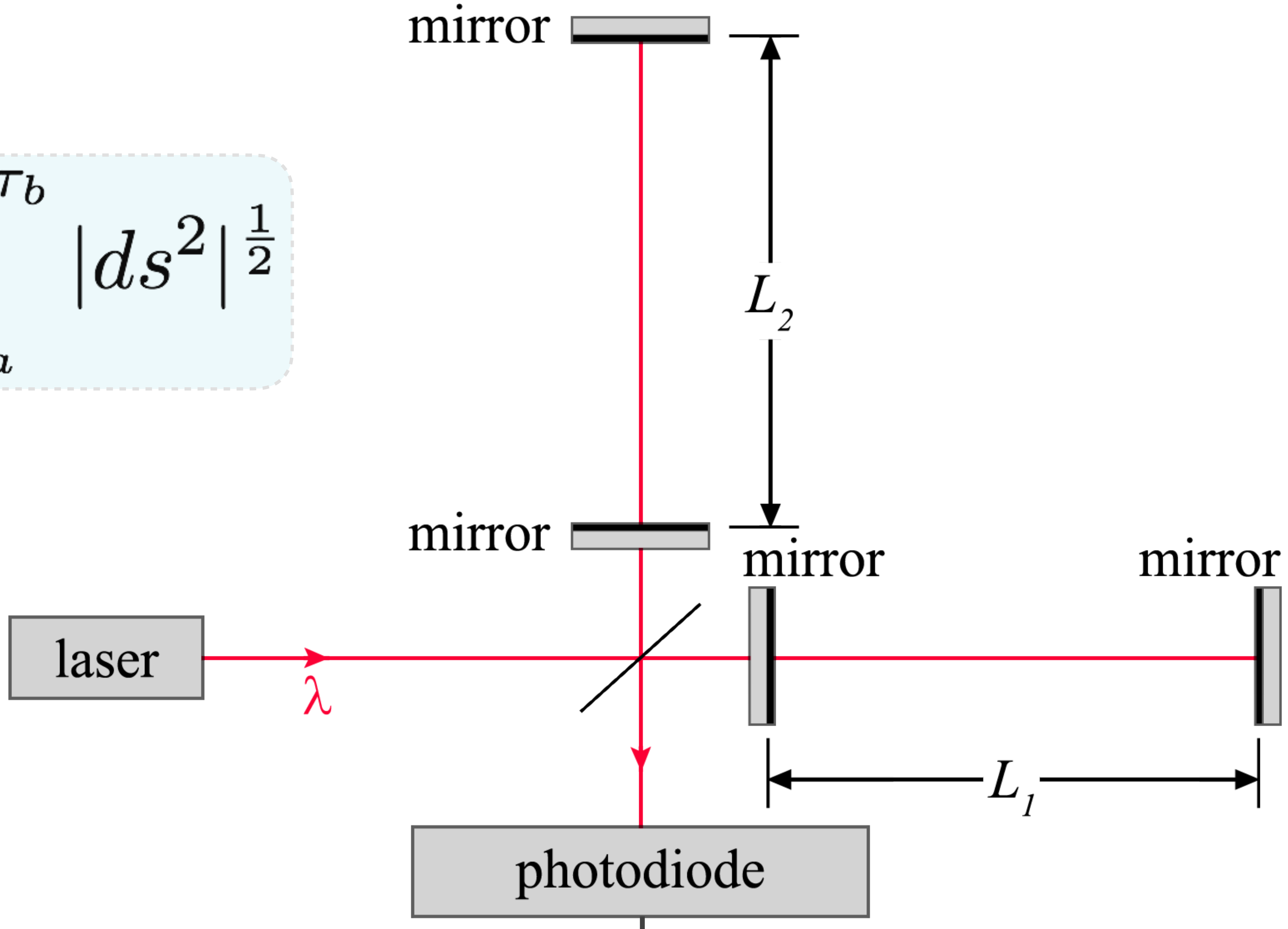
Radiative Multipole Moment: $h_{\ell m}(t)$


h_+


$h_\times$

From Einstein to the first gravitational wave detection

$$L = \int_{\tau_a}^{\tau_b} |ds^2|^{\frac{1}{2}}$$



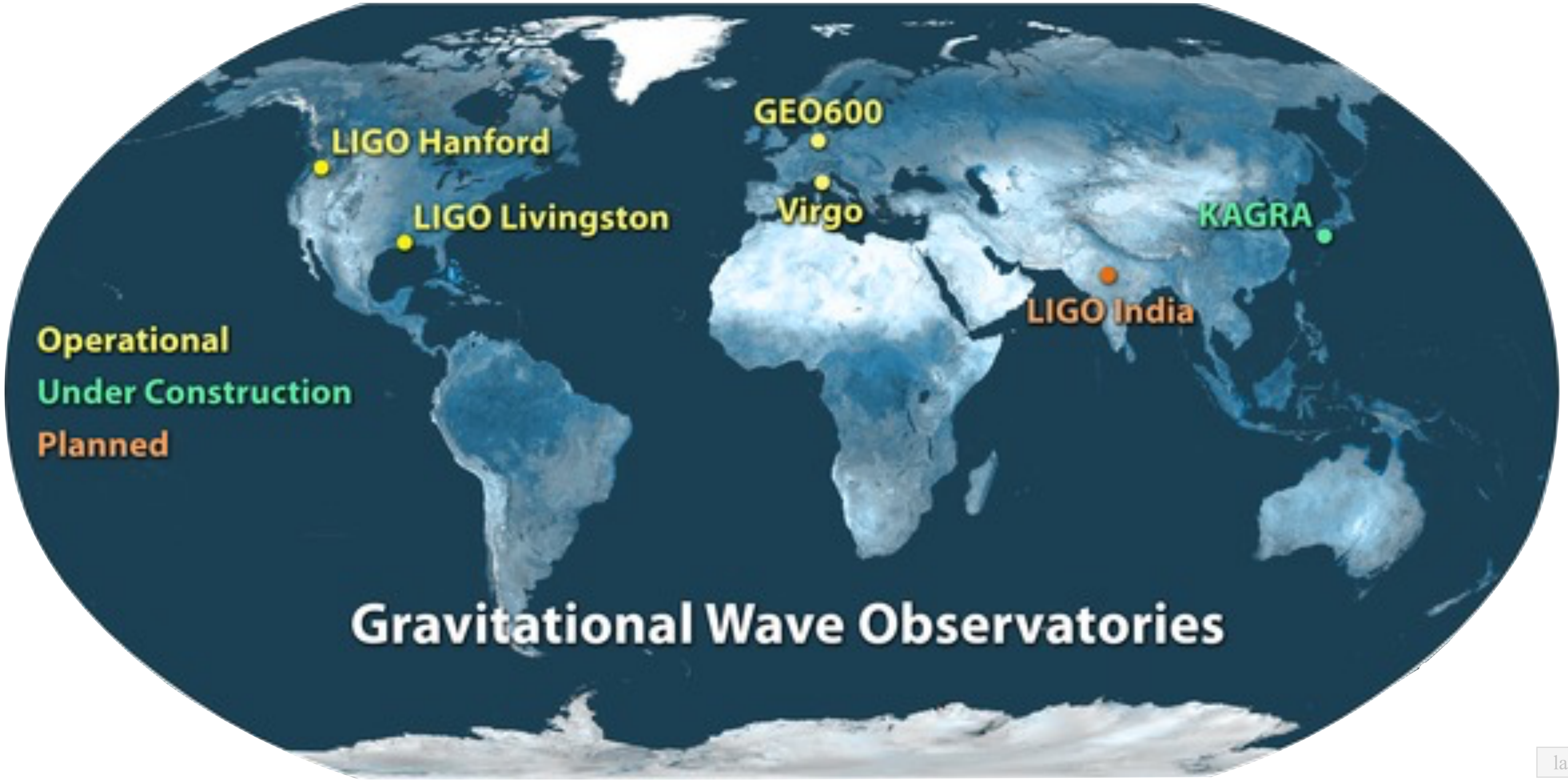
$$h_{\text{Response}} = F_+ h_+ + F_\times h_\times$$

1972 Rainer Weiss
Proposed interferometric GW detectors

1994 LIGO Construction
Construction begins after some initial turmoil

2002 LIGO begins collecting data
Data analysis begins

From Einstein to the first gravitational wave detection



Gravitational wave Observatories as of **2018**

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Analysis development
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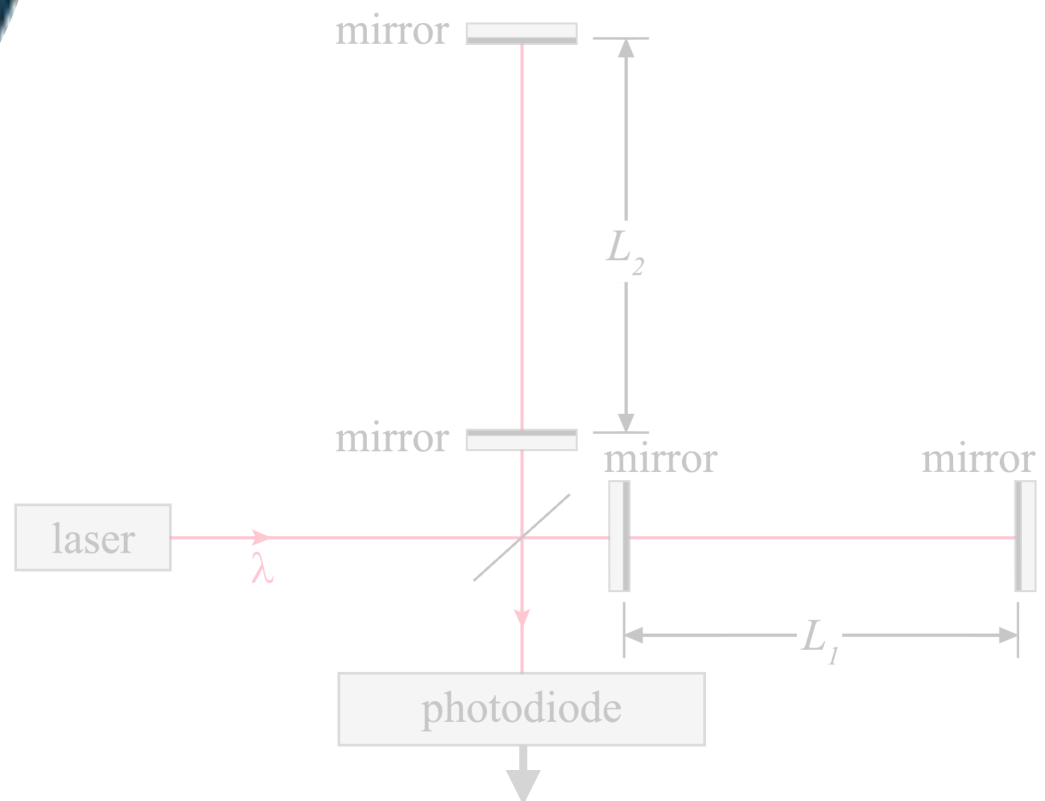

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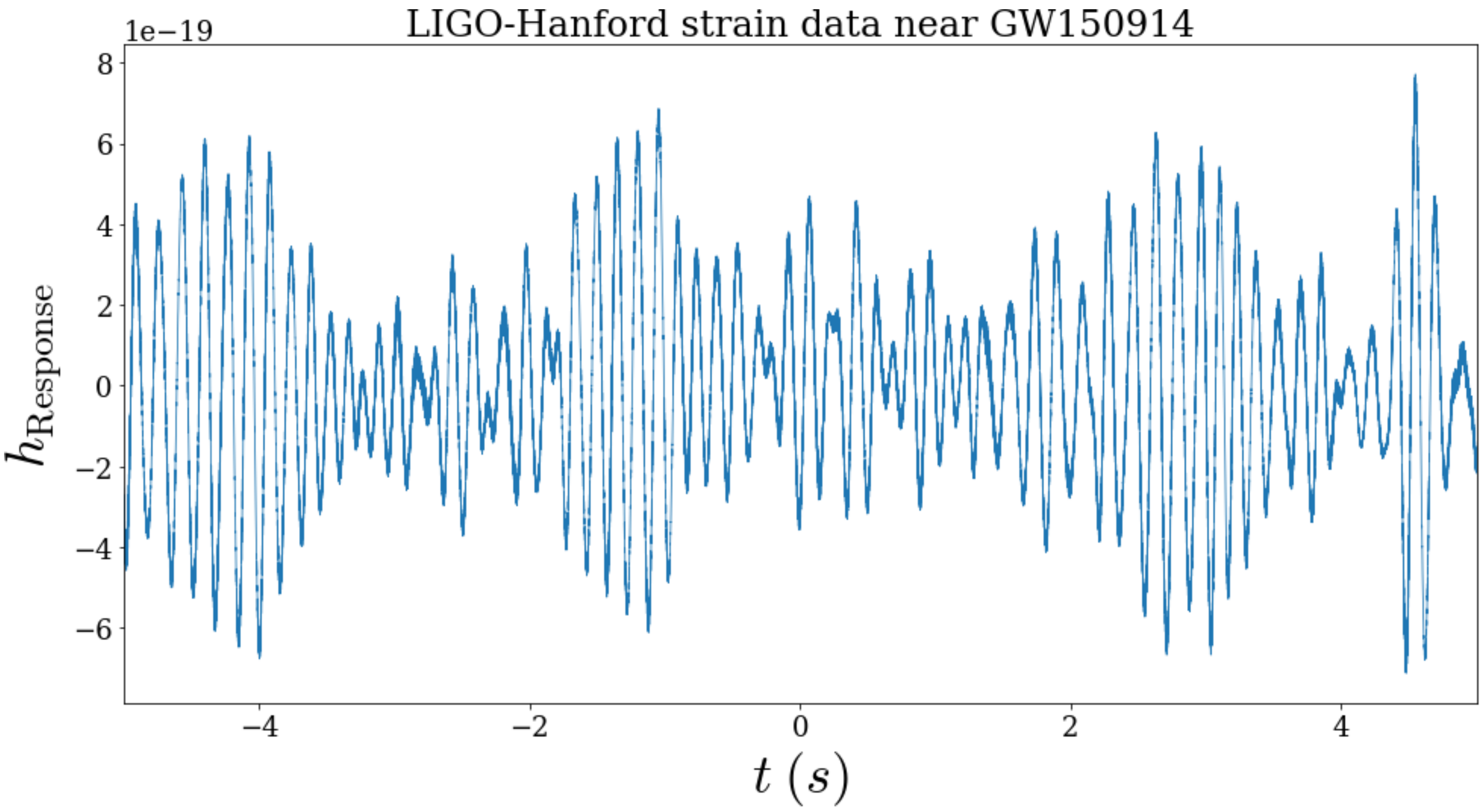
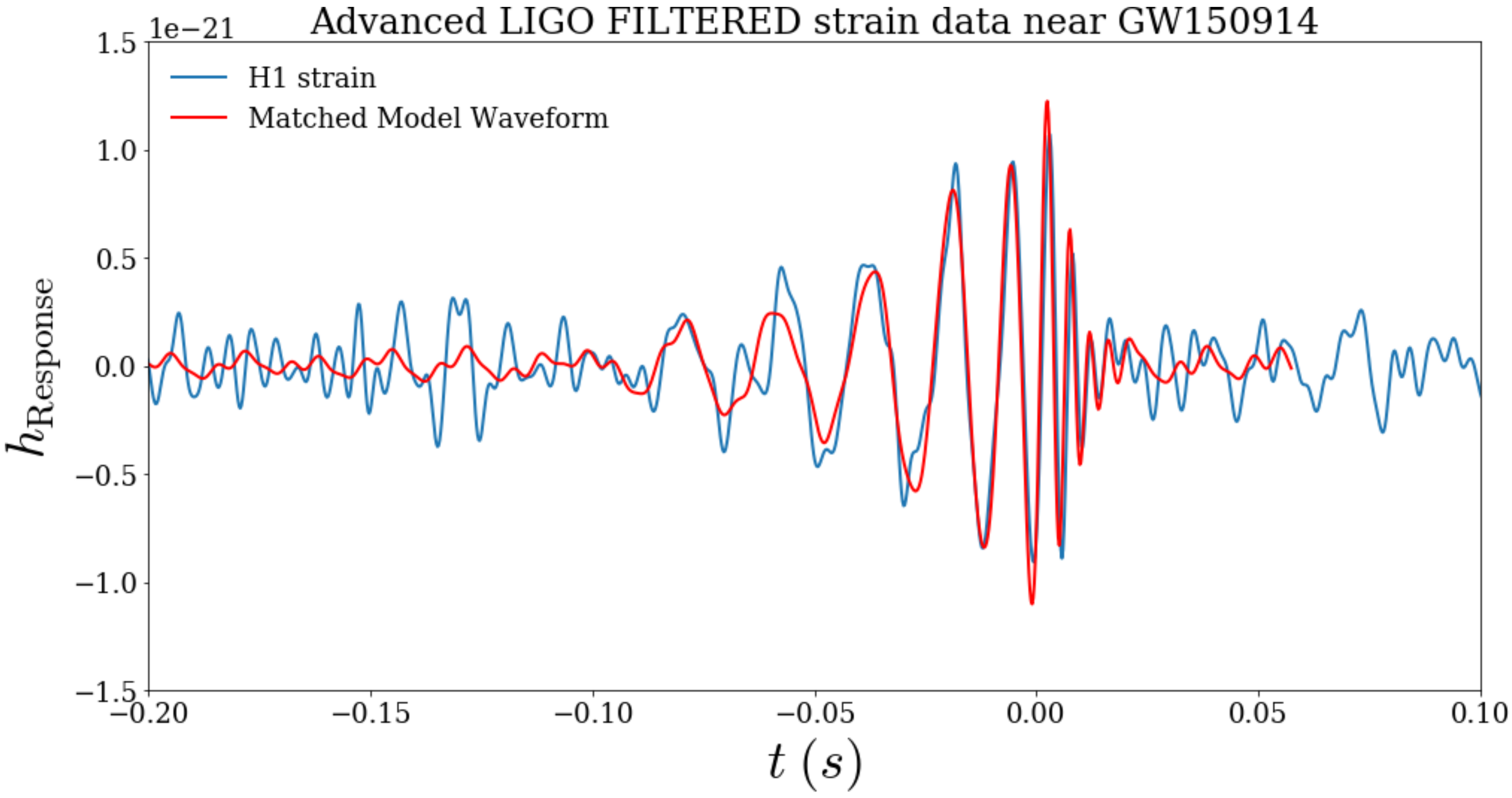
From Einstein to the first gravitational wave detection



Thankfully, we know how to eliminate known noise sources from raw data.

From Einstein to the first gravitational wave detection

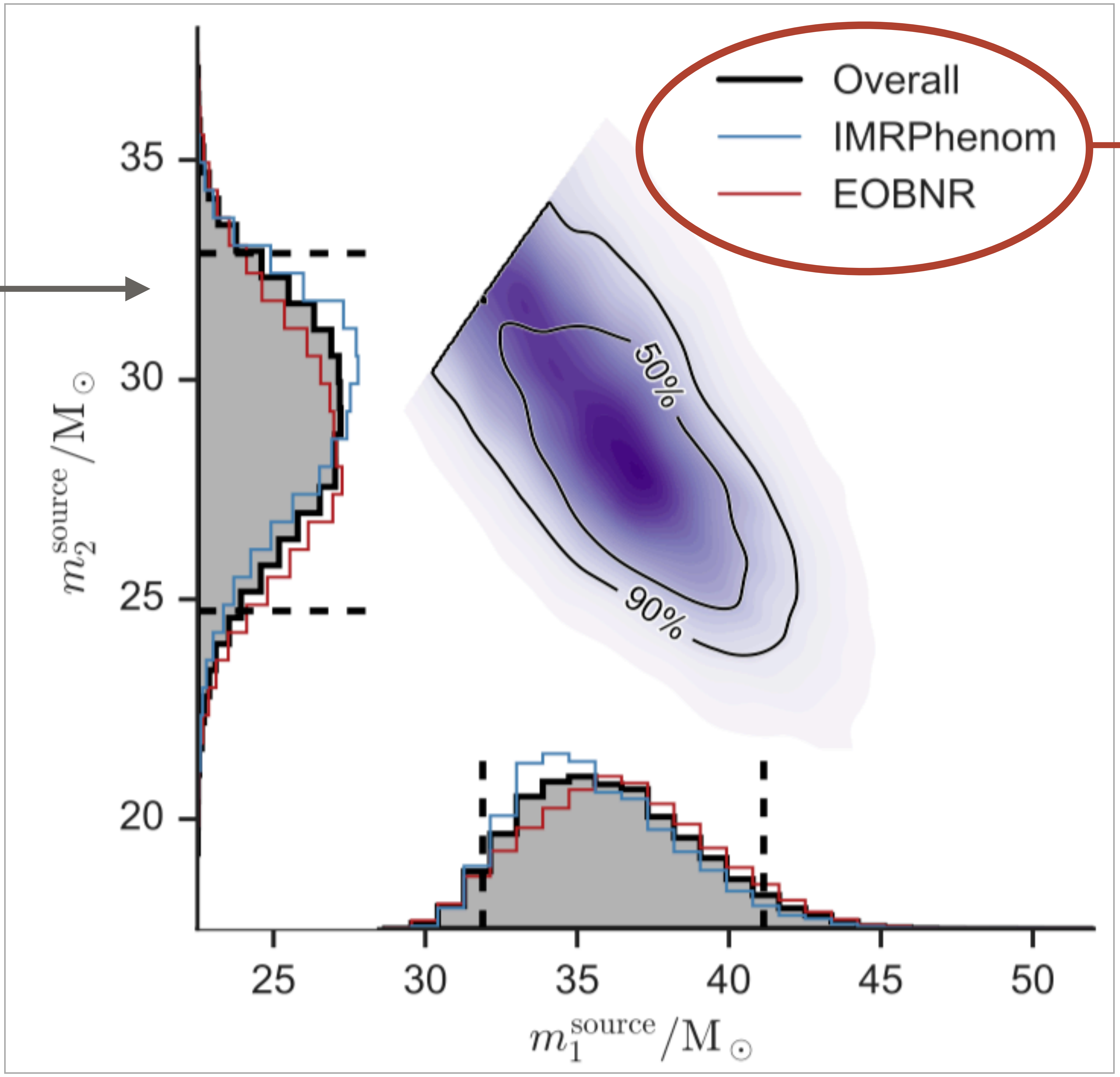


We know how "fit" experimental data with signal models

From Einstein to the first gravitational wave detection

A "**posterior distribution**" —
A histogram whose height indicates the likelihood of e.g. a BH mass taking on a certain value

Uncertainty in inference is due to detector noise (sensitivity), relative signal loudness (**SNR**), and **error in signal models** …



GW Signal Models enable us to connect data from experiment, with fundamental physics, and thereby output astrophysical relevant information, such as BH masses…

And we know how to interface signal models with statistical inference to estimate source parameters

Primary Challenges

- **Develop robust analytic descriptions of non-Hermitian physics** (e.g. black hole modes) within Einstein's GR: Focus on most observable affects that can be precisely understood.
- Develop **practical tools to <u>interface</u> theory, simulations, and data analysis**.

Preface: a few notes on my background

- *2014* "Modeling Ringdown: Beyond the Fundamental Quasi-Normal Modes" London, Healy, Shoemaker
  - First ringdown signal model to include overtones, 2nd order modes
  - Prediction the *l=m=3* moment would be detectable after (2,2)
- *2017* "First higher-multipole model of gravitational waves from spinning and coalescing black-hole binaries" London, Khan, *et. al.*
  - Used in data analysis for GW190412, the first detection of higher harmonics.
  - Derived techniques still used in state of the art "PhenomX" models
- *2020* "Bi-orthogonal harmonics for the decomposition of gravitational radiation I/II" London+Hughes
  - Introduction of "adjoint spheroidal harmonics"
  - First theory and applications of QNM "bi-orthogonality" in angular sector

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- *2022* "Bi-orthogonal harmonics for the decomposition of gravitational radiation I/II" London+Hughes
  - Introduction of "adjoint spheroidal harmonics"
  - Premise: we can do better than fitting — QNM projection is promising

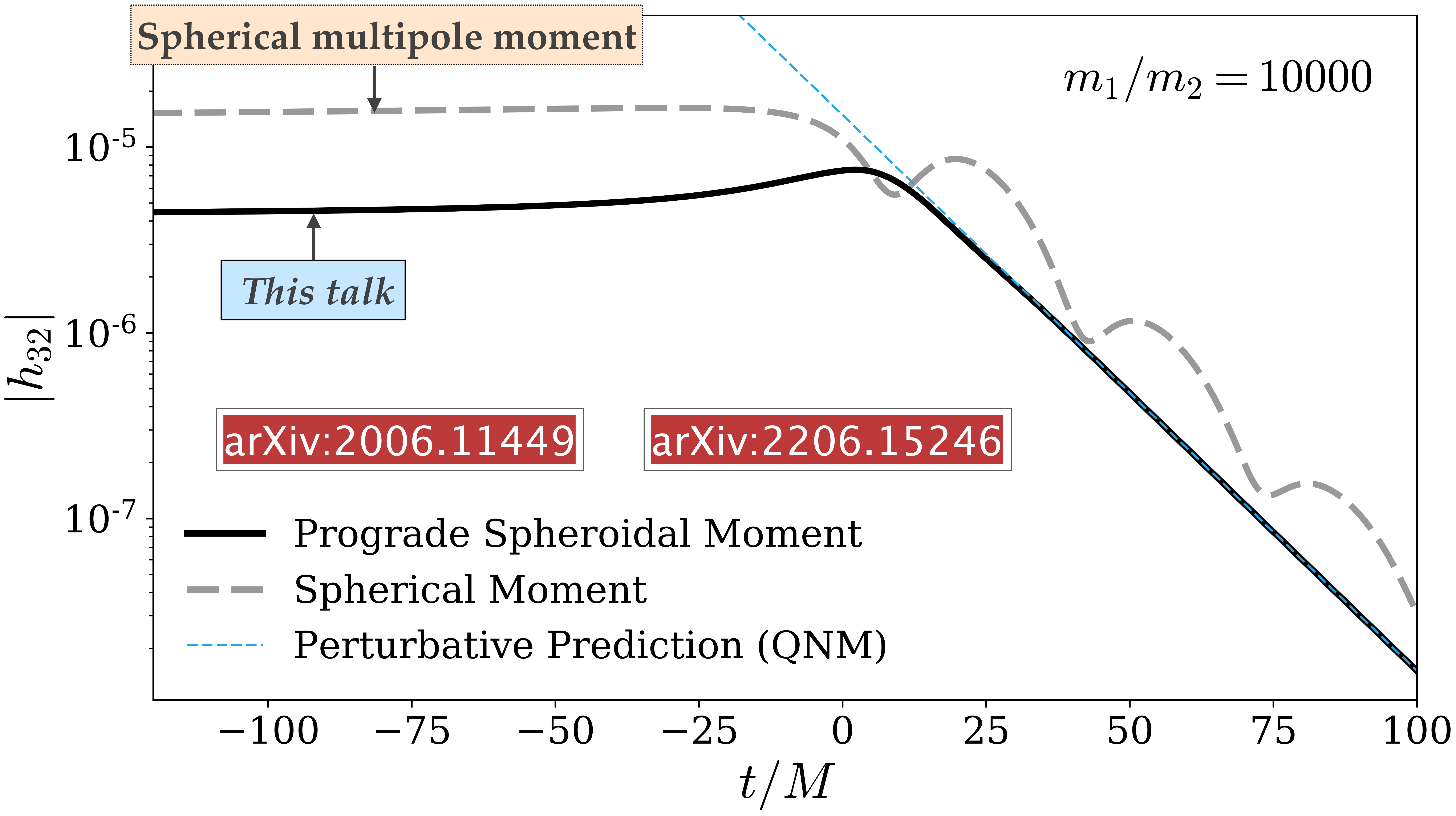
Preface: we can do better …

- There is good reason to think that we still lack a first principles understanding of binary black hole (BBH) merger. Without this understanding, what we can learn for astrophysical BBH signals is limited.
- Moreover, we still (arguably) have a limited understanding of perturbed single black holes — it is entirely likely that we cannot better understand merger without a more complete understanding of the perturbed remnant black hole.
- A central idea of this talk is that a deeper understanding of Einstein's equations linearly perturbed around Kerr may enable us to learn more about not only BBH ringdown, but also more general scenarios.

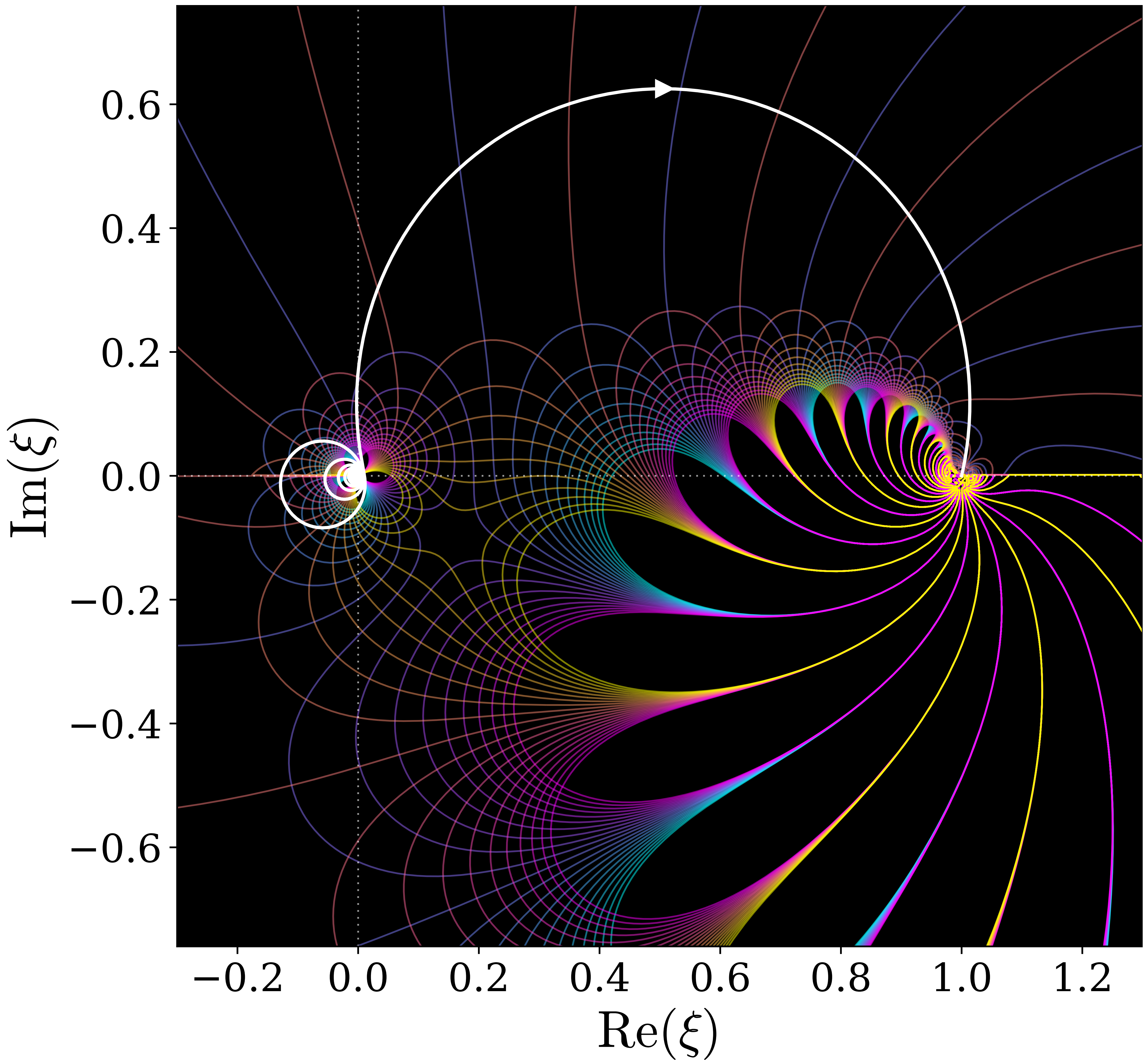
Preface: we can do better …

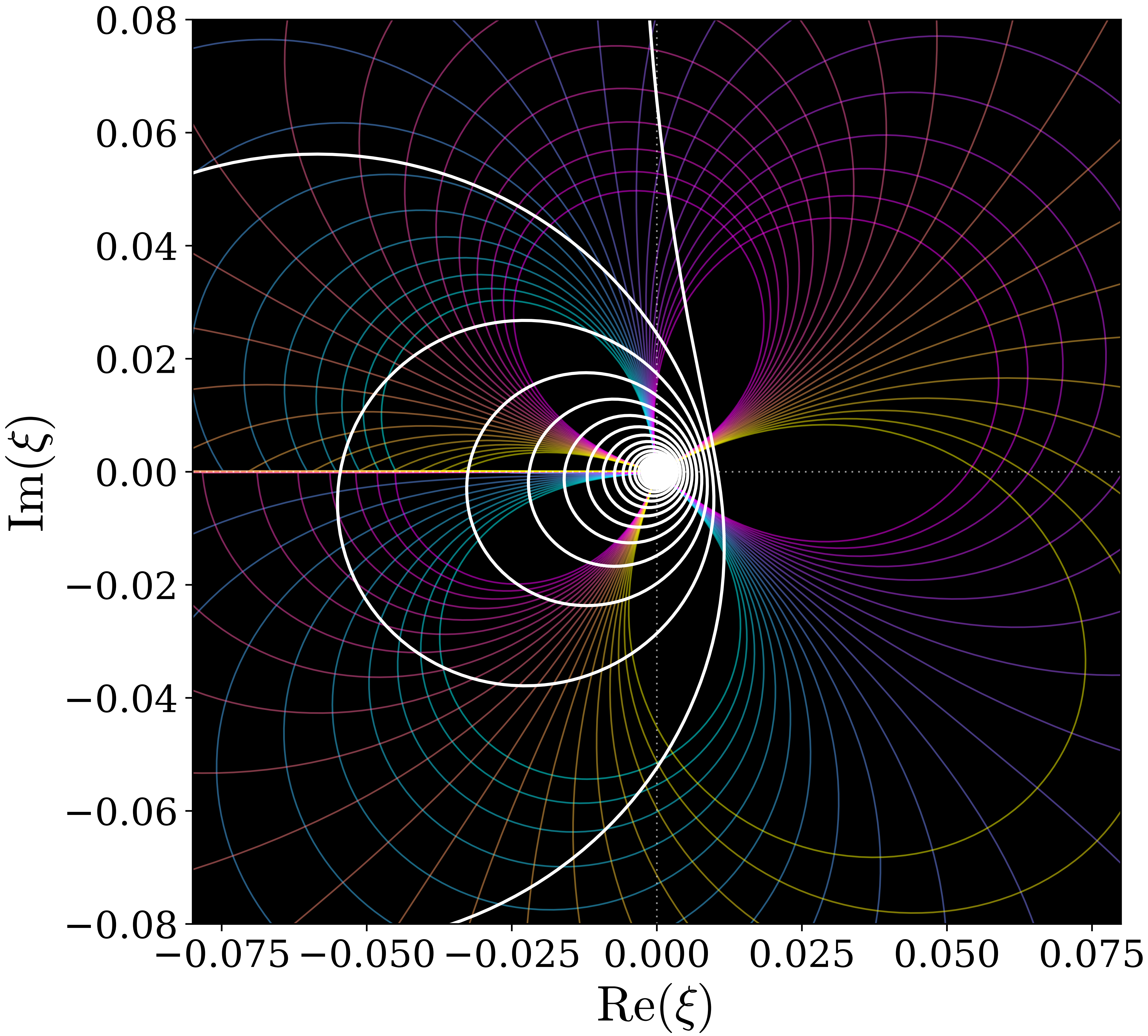
- Linear perturbations of Kerr have two non-trivial dimensions (in Boyer-Lindquist coordinates) — these are radius and polar angle.
- (**1**) A better understanding of the angular problem points the way to a new representation of GWs from arbitrary sources.
- (**2**) A better understanding of the radial problem enables a deeper understanding of so-called ringdown "overtones" and thereby pointing to new techniques and results.
- <u>Basic technical idea</u>: Each problem can be treated independently, and then combined. There is fun (i.e. learning) in how this is more complex than it sounds after one applies Sturm-Liouville theory.

This talk: we can better represent numerical GWs

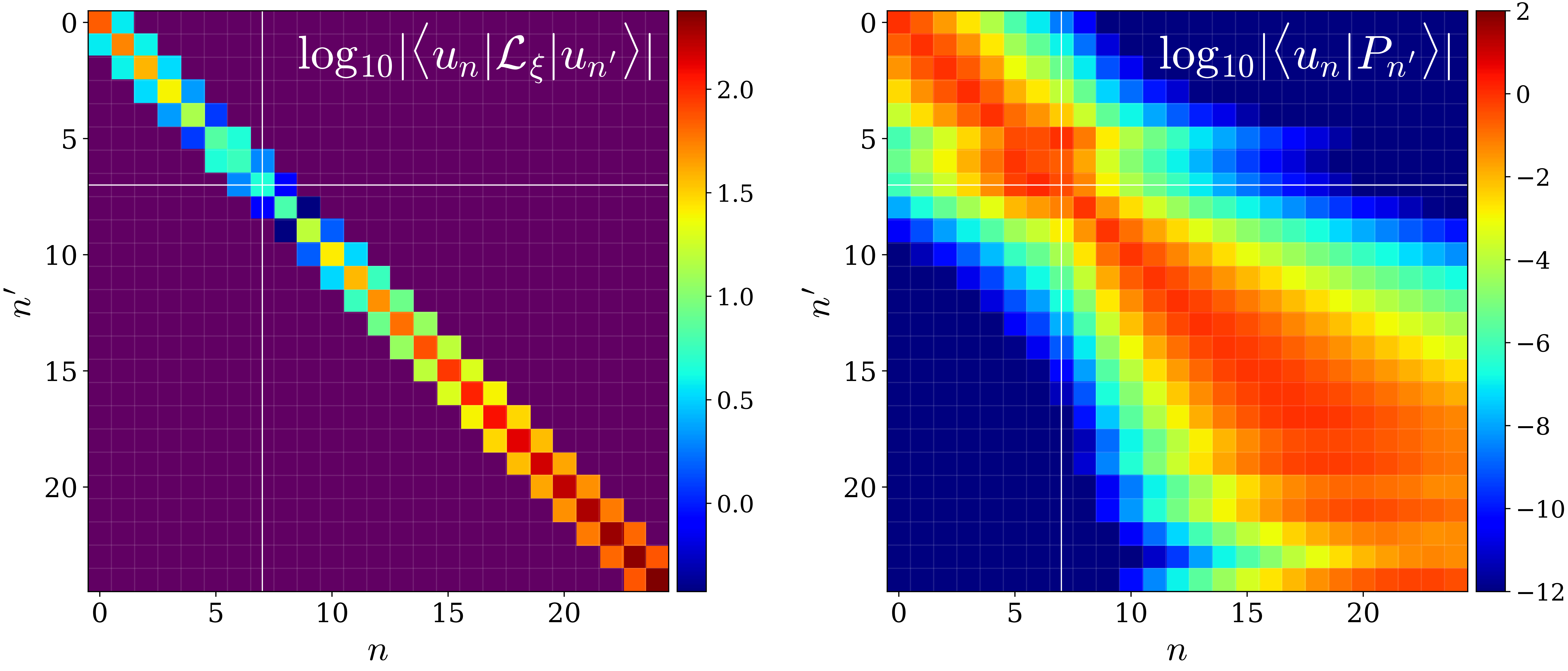


This talk: we can better understand radial scalar products

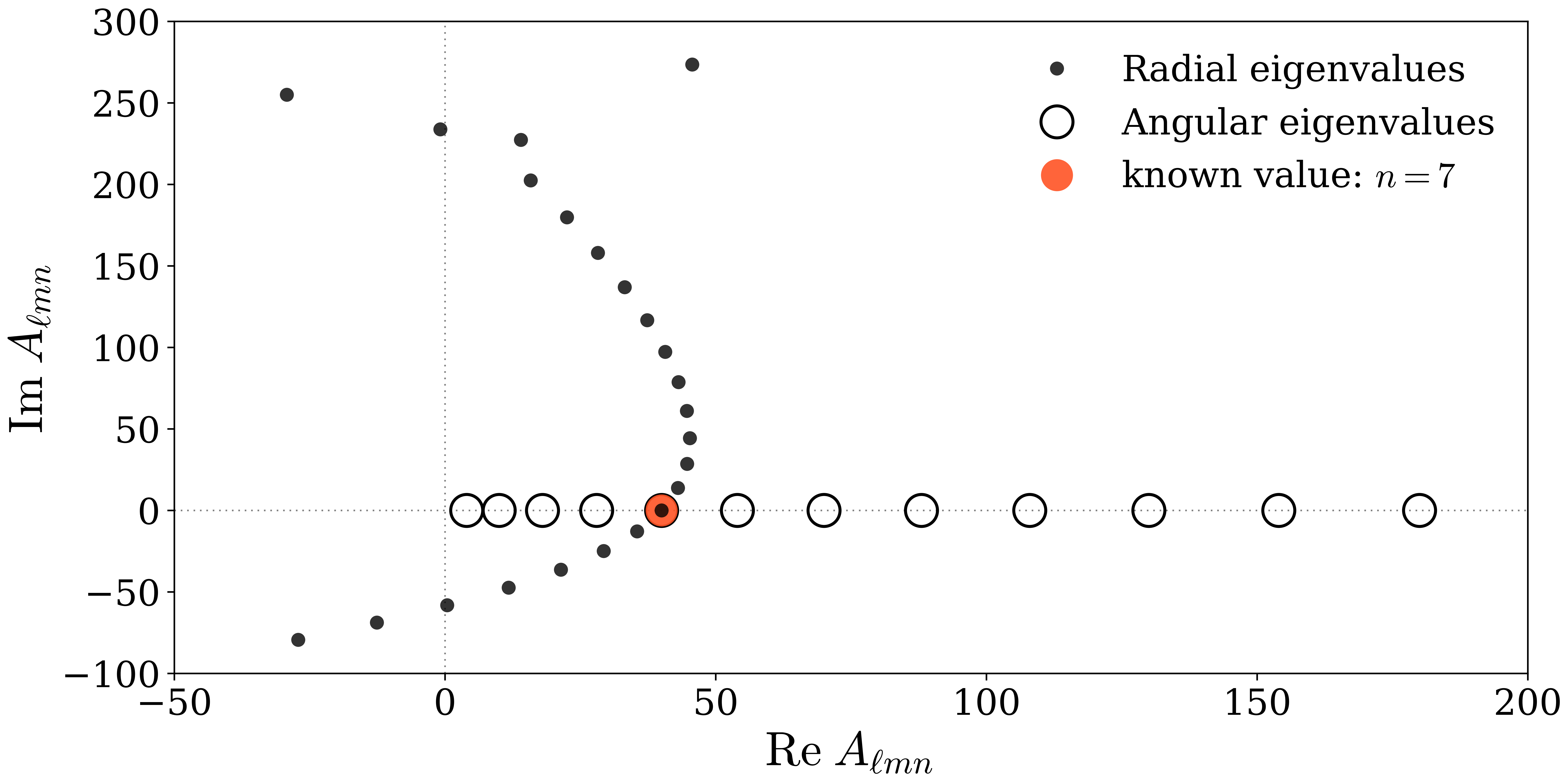




This talk: we can better represent the radial problem



This talk: we can better understand "overtones"



Foundational concepts Linear perturbations of Kerr

A conceptual tension

There is a perhaps a conceptual tension: Should we think of the BBH merger signals as something strongly nonlinear relative to inspiral?

A conceptual tension

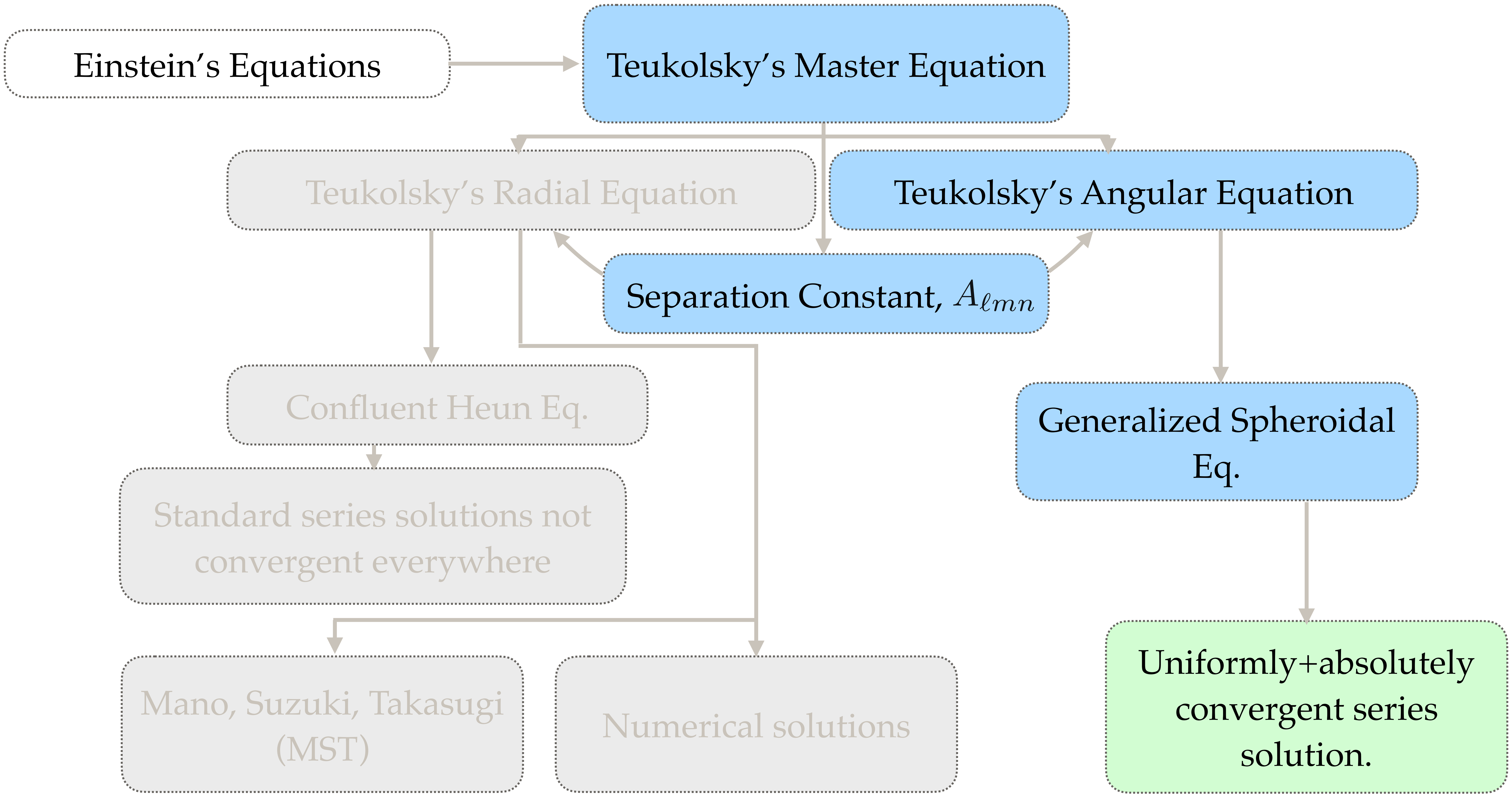
OR Should we think of BBH merger signals as being, essentially or approximately, QNM ringdown (i.e. linear perturbations of Kerr)?

An interpretation of the tension

- Many of the reasons for this and related tensions are straightforward:
  - There is perhaps an impression that QNMs are primarily about time domain ringdown (e.g. damped sinusoids). This isn't quite accurate. (e.g. Teukolsky '71, Leaver '85 and many others)
  - The field equations for linearly perturbed BHs are non-hypergeometric and parametrically coupled, thus much of the intuition we've learned from elementary quantum mechanical systems does not apply …
  - In this sense, we really don't understand perturbed (classical) Kerr BHs with the same depth with which we understand e.g. hydrogen atoms …

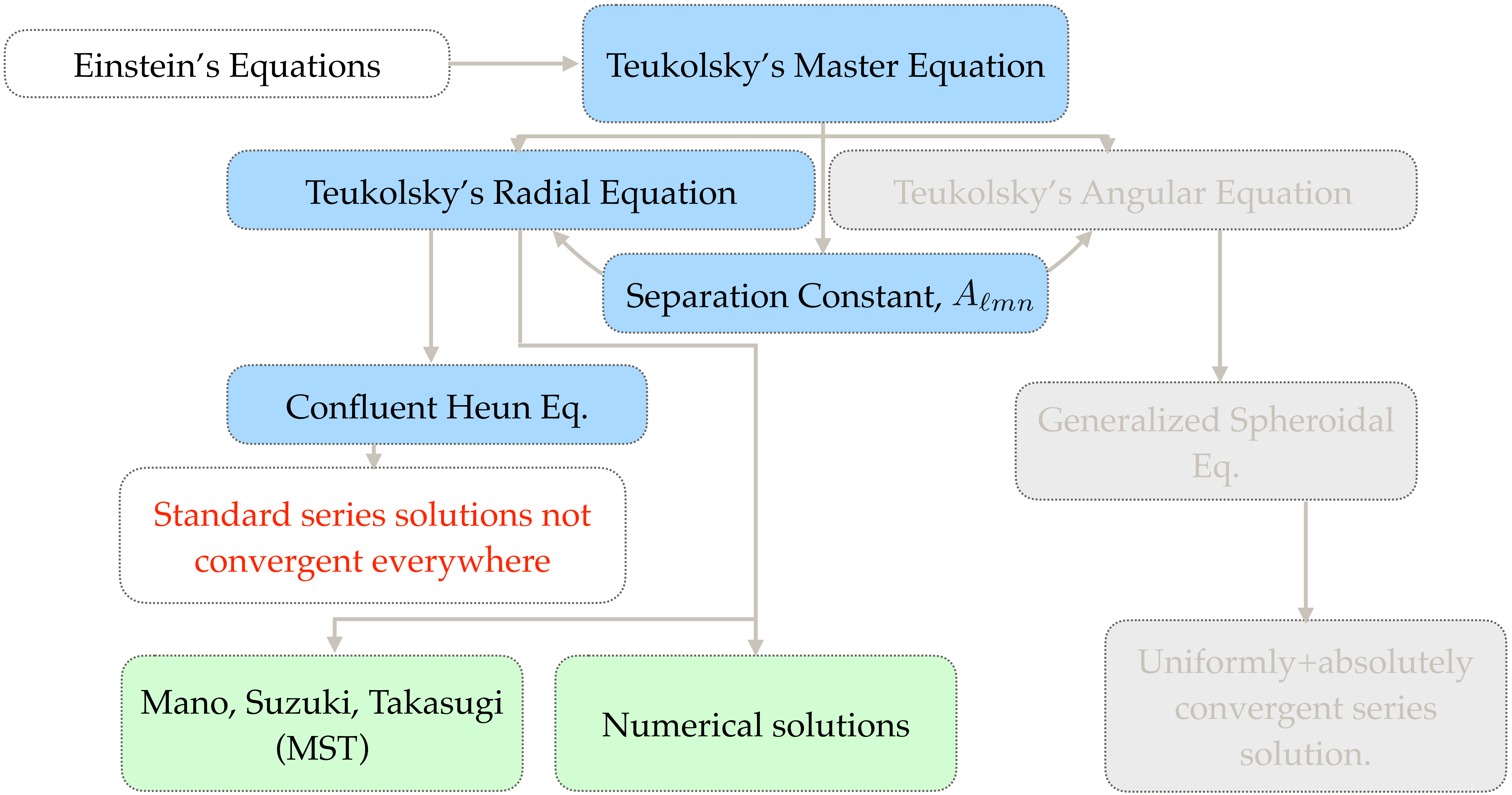
Linearly perturbed Kerr BHs pose an interesting technical problem …

Outline of the problem



$$\mathcal{L}_\theta \, S(\theta) \; = \; -A \, S(\theta)$$

Outline of the problem



$$\mathcal{L}_r \; P(r) \; = \; +A \; P(r)$$

The current state of affairs (a biased outline)

- There is increasing evidence that the radial and angular functions are orthogonal in some way. (e.g. London 2020, Green+ 2022, Ma+2024)
- I have recently shown that the angular functions (the spheroidal harmonics) are complete and bi-orthogonal. (London 2020, London+Hughes '22)
- Green+, Ma+ and others have shown a kind of spatial orthogonality, but full spatial completeness remains an open question. (Green+ 2022,Ma+2024)
- Recent work on the mathematical nature of Teukolsky's radial equation reveals that its eigenfunctions are typically complete or *over*-complete. (London, London+Gurevich 2024)

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A spheroidal picture

for GWs from arbitrary sources

arXiv:2006.11449

arXiv:2206.15246

The hard parts and their solutions

The hard parts

- Each spheroidal harmonic with labels ***(l,m,n)*** is the eigenfunction of a single differential equation — since there is an infinity of such labels, there is an infinity of differential operators that must be taken into account. As a result, the QNMs spheroidal harmonics are not orthogonal …
- Each differential equation corresponds to a differential operator that is ****almost**** hermitian

Their solutions

- One approach is to use linear forms (i.e. construct a single operator (via projection) for which all spheroidal harmonics are eigenfunctions)
- This is underpinned by the notion that the spheroidal harmonics can be uniquely and reversibly transformed into spherical harmonics
- This leads to a new sequence of special functions — "**adjoint-spheroidal harmonics**"

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The hard parts and their solutions

The hard parts

Teukolsky's angular operator

- $$\mathcal{D}_u(\omega_{\ell mn}) = (ua\omega_{\ell mn} - 2s)ua\omega_{\ell mn} - \frac{(m+su)^2}{1-u^2} + \partial_u(1-u^2)\partial_u$$

- $$\mathcal{D}_u(\omega_{\ell mn})^{\dagger} = \mathcal{D}_u(\omega_{\ell mn})^{*}$$

Their solutions

- $$\mathcal{T} = \sum_{\ell=2}^{\infty}\sum_{\ell'=2}^{\infty} |Y_{\ell m}\rangle\langle Y_{\ell m}|S_{\ell' m}\rangle\langle Y_{\ell' m}|$$

- $$|S_\ell\rangle = \mathcal{T}\,|Y_{\ell m}\rangle\,, \quad \text{and} \quad |\tilde{S}_\ell\rangle = \mathcal{T}^{\dagger^{-1}}|Y_\ell\rangle$$

new "adjont-spheroidals"

The hard parts and their solutions

The hard parts

Teukolsky's angular operator

- $$\mathcal{D}_u(\omega_{\ell mn}) = (ua\omega_{\ell mn} - 2s)ua\omega_{\ell mn} - \frac{(m+su)^2}{1-u^2} + \partial_u(1-u^2)\partial_u$$

- $$\mathcal{D}_u(\omega_{\ell mn})^\dagger = \mathcal{D}_u(\omega_{\ell mn})^*$$

Their solutions

bi-orthogonality

- $$\langle \tilde{S}_\ell | S_{\ell'} \rangle = \int_\Omega {}_s\tilde{S}_\ell(\theta; \tilde{\omega}_{\ell mn})\ {}_sS_\ell(\theta; \tilde{\omega}_{\ell mn})\ d\Omega = \delta_{\ell\ell'}/2\pi$$

- $$\sum_{\ell=2}^{\infty}\sum_{\ell'=2}^{\infty} |S_\ell\rangle\langle\tilde{S}_{\ell'}| = \hat{\mathbb{I}}$$

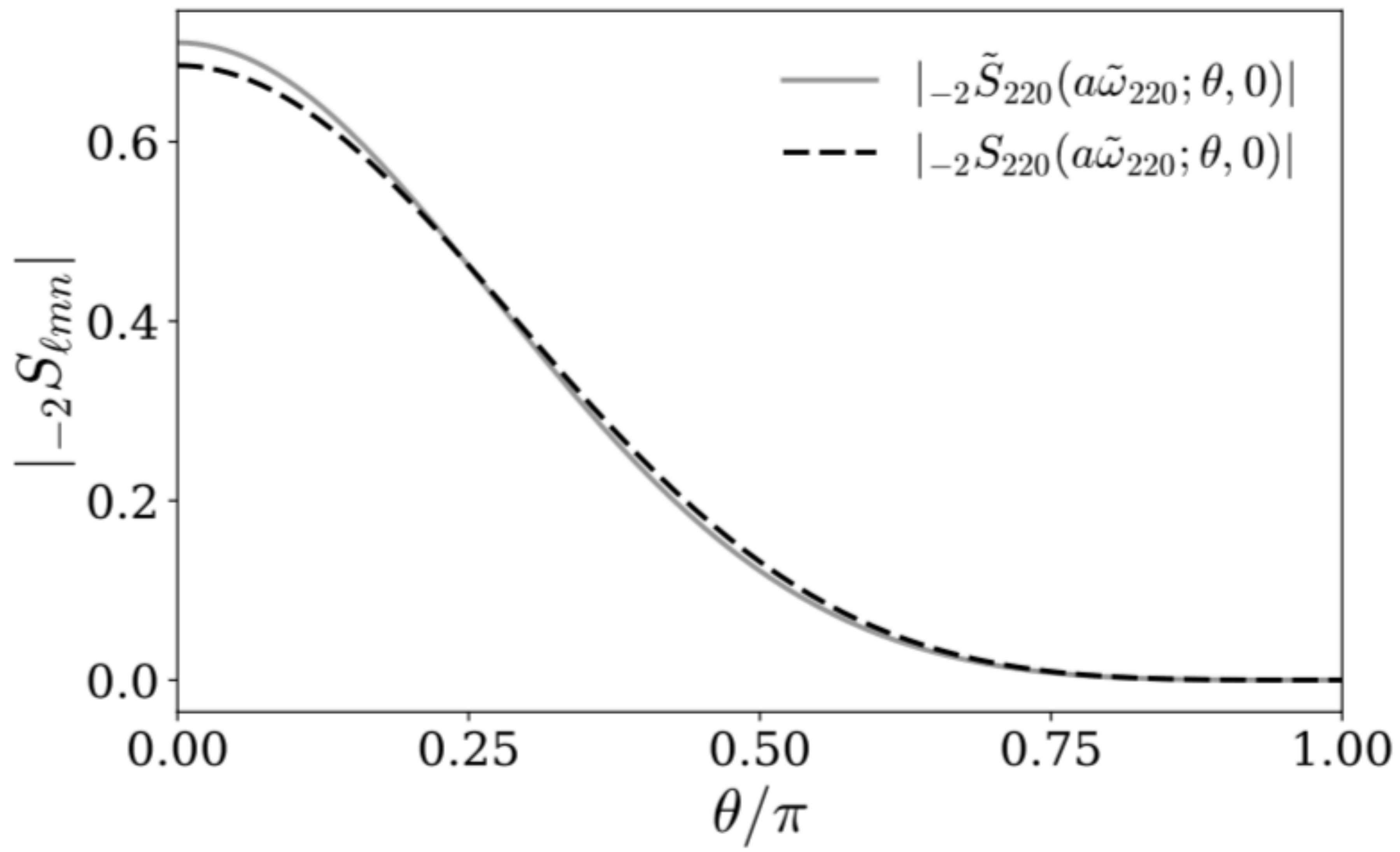
completeness: the ability to represent arbitrary GW signals

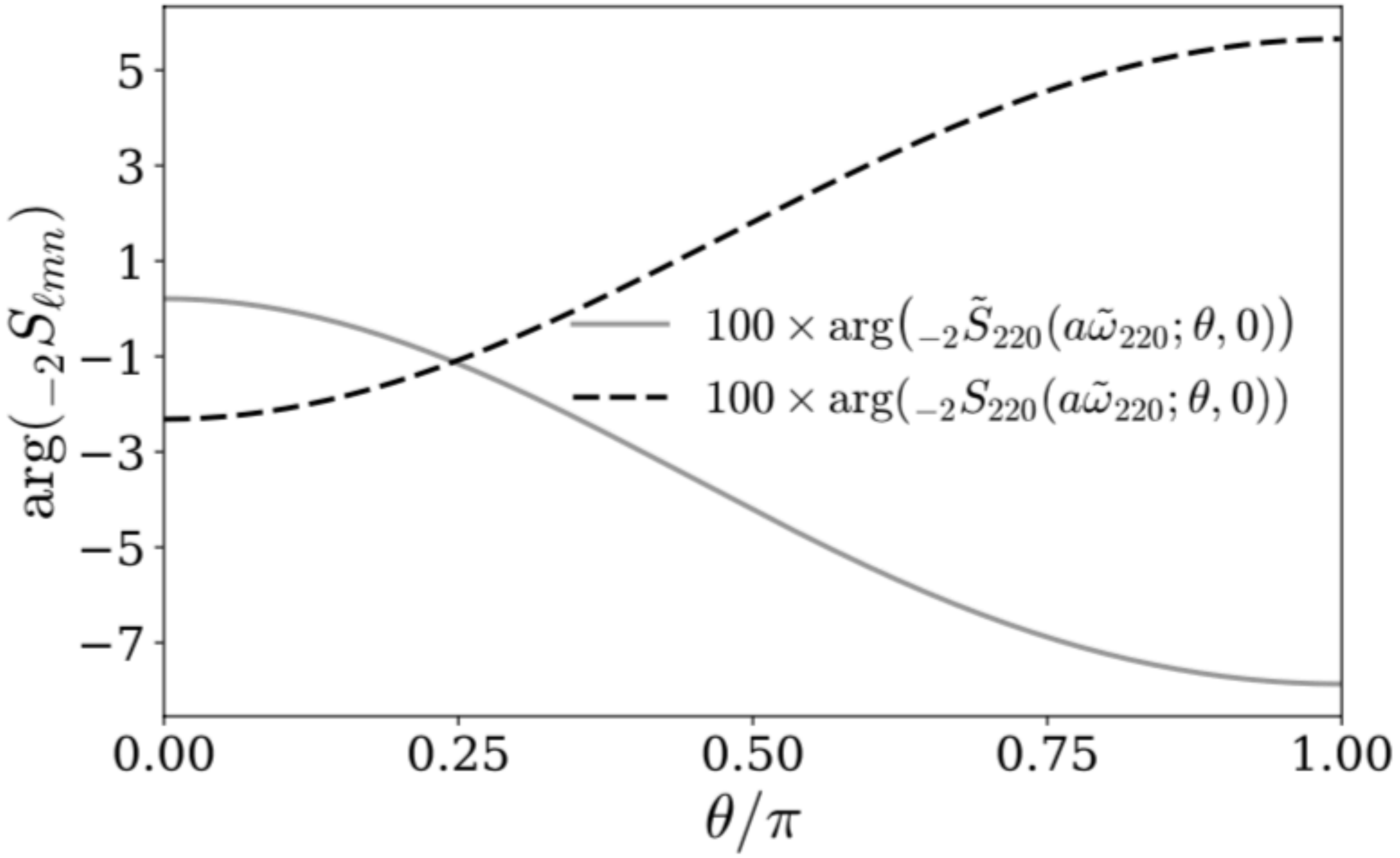
Some intuition about Adjoint-Spheroidal Harmonics

Example “small spin” expansion for term-by-term comparison:

$$S_{\ell mn} \approx \; Y_{\ell m} + a\tilde{\omega}_{\ell mn} c_{\ell}^{\ell-1} Y_{\ell-1,m} + a\tilde{\omega}_{\ell mn} c_{\ell}^{\ell+1} Y_{\ell+1,m}$$

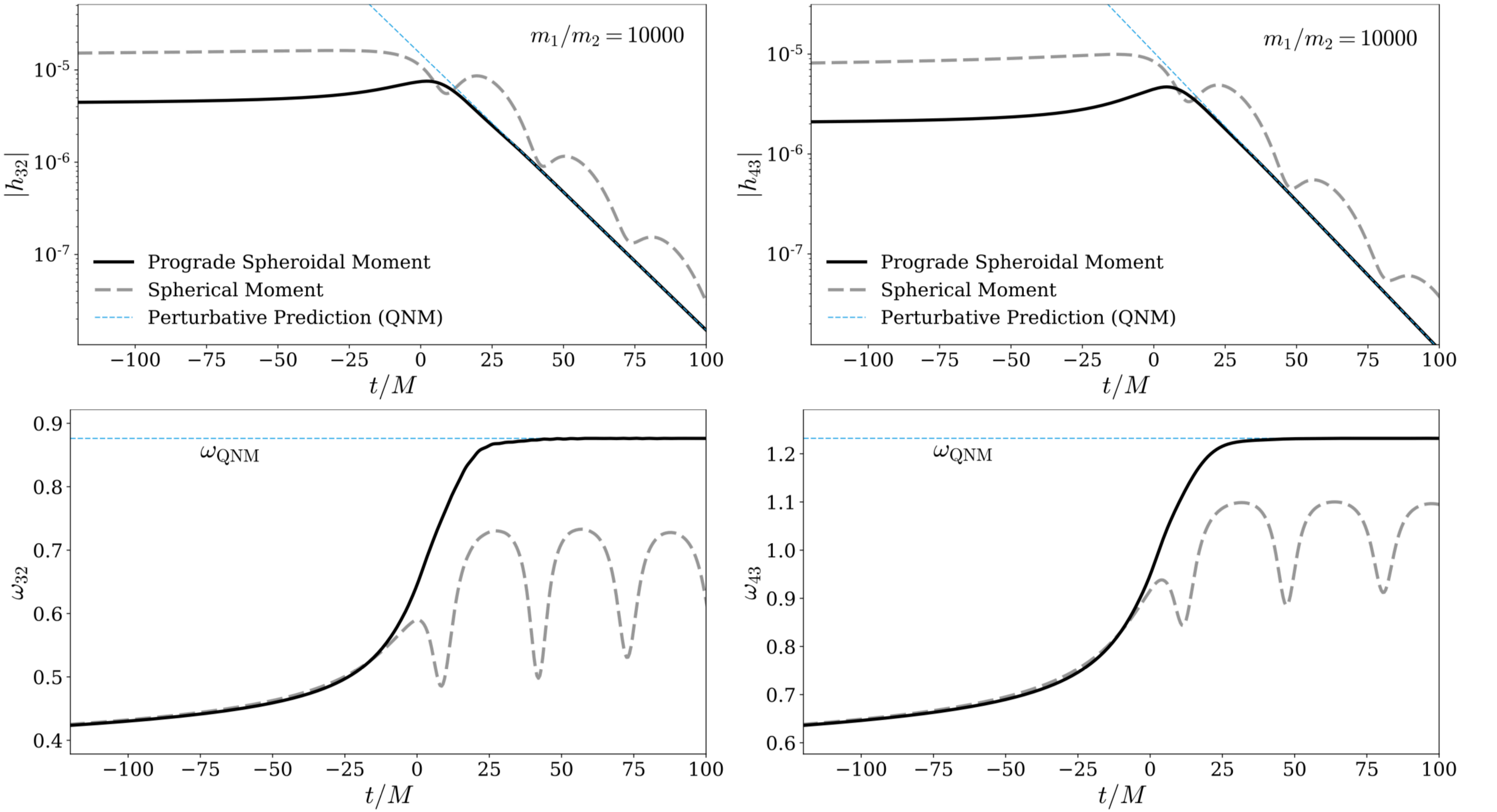
$$\tilde{S}_{\ell mn} \approx -Y_{\ell m} + a\tilde{\omega}^*_{\ell-1,m,n} c^{\ell}_{\ell-1} Y_{\ell-1,m} + a\tilde{\omega}^*_{\ell+1,m,n} c^{\ell}_{\ell+1} Y_{\ell+1,m}$$





London+Hughes, arXiv:2206.15246

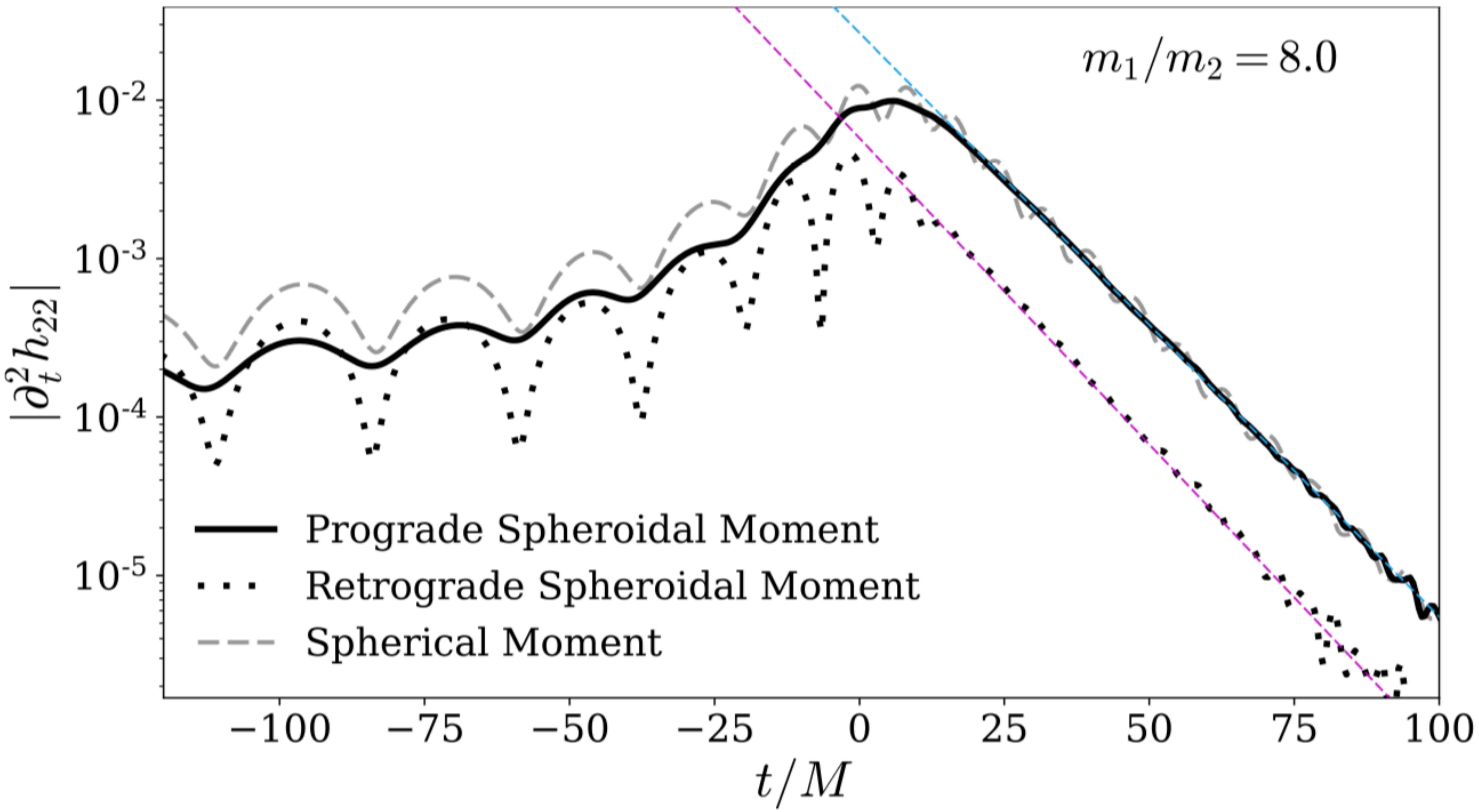
Example applications: Extreme Mass-Ratio Binary



London+Hughes, arXiv:2206.15246

Example applications: Comparable Mass-Ratio Binary

Mass ratio 8:1, **precessing** BBH simulation (BAM, Cardiff University)



An exact tri-diagonalization of Teukolsky's radial equation for QNMs

arXiv:2312.17678

arXiv:2312.17680

A purely “spectral” approach to Teukolsky’s radial equation for QNMs

arXiv:2312.17678

arXiv:2312.17680

Some quirks of the radial problem

- Many studies apply special functions from other problems. Here we focus on and develop special functions motivated directly by Teukolsky's radial problem
- We will draw as much as possible from (non-Hermitian) Sturm-Liouville theory. For this, a suitable coordinate choice is very useful
- For QNMs, we are interested in the "exterior problem" — i.e. spacetime between the event horizon and spatial infinity

Let's go through this step-by-step …

$$\mathcal{L}_r \, R(r) \;=\; A \, R(r)$$

For QNMs, we seek to solve an eigenvalue problem, where the eigenvalue is the separation constant.

Teukolsky's radial equation

Use fractional radial coord

$$\mathcal{L}_r = \left(\mathrm{A}_0 + r\mathrm{A}_1 + (\mathrm{A}_2\, r)^2 + \frac{\mathrm{A}_3}{r - r_-} + \frac{\mathrm{A}_4}{r - r_+} \right) + (\mathrm{A}_5 + \mathrm{A}_6 r)\, \partial_r + (r - r_-)(r - r_+)\, \partial_r^2$$

The related differential operator is not well formatted for the exterior problem ….

Teukolsky's radial equation | Use fractional radial coord

$$\xi = \frac{r - r_+}{r - r_-}$$

$$\mathcal{L}_\xi = \left(\mathrm{B0} + \frac{\mathrm{B1}}{\xi} + \frac{\mathrm{B2}}{1-\xi} + \frac{\mathrm{B3}^2}{(1-\xi)^2} + \mathrm{B4}(1-\xi) \right) + \left(\mathrm{B5}(1-\xi) + \mathrm{B6}(1-\xi)^2 \right) \partial_\xi + \xi(1-\xi)^2 \partial_\xi^2$$

One approach taken by Leaver in 1985 was to use what I'll call a "**fractional radial coordinate**"

ukolsky's radial equation | Use fractional radial coordinates | Apply QNM boundary co

$$R(r(\xi)) = \mu(\xi) f(\xi)$$

$$\mu(\xi) = e^{\frac{2i\delta\tilde{\omega}}{1-\xi}} (1-\xi)^{1+2(s-iM\tilde{\omega})} \times \xi^{-(iM\tilde{\omega}+s)+\frac{i\left(am-2M^2\tilde{\omega}\right)}{2\delta}}$$

From here we can apply the QNM boundary conditions, which amount to a **similarity transformation** of the problem …

actional radial coordinates | **Apply QNM boundary conditions** | Study the radial OD

$$\left[\mu(\xi)^{-1}\,\mathcal{L}_r\,\mu(\xi(r))\right] f(\xi) \;=\; \left[\mu(\xi)^{-1} A\,\mu(\xi)\right] f(\xi)$$

$$\mathcal{L}_\xi\, f(\xi) \;=\; A\, f(\xi)$$

From here we can apply the QNM boundary conditions, which amount to a **similarity transformation** of the problem …

actional radial coordinates | **Apply QNM boundary conditions** | Study the radial OD

$$
\begin{aligned}
\mathcal{L}_\xi &= \left(\mathrm{C}_0 + \mathrm{C}_1(1-\xi)\right) \\
&+ \left(\mathrm{C}_2 + \mathrm{C}_3(1-\xi) + \mathrm{C}_4(1-\xi)^2\right)\partial_\xi + \xi(\xi-1)^2\partial_\xi^2
\end{aligned}
$$

The resulting differential operator has a regular potential, and its second derivative coefficient **simplifies boundary condition requirements** of Sturm-Liouville theory

actional radial coordinates | **Apply QNM boundary conditions** | Study the radial OD

$$\langle \mathrm{a} \,|\, \mathrm{b} \rangle \;=\; \int_0^1 \mathrm{a}(\xi)\, \mathrm{b}(\xi)\;\; \xi^{\mathrm{B}_0'} (1-\xi)^{\mathrm{B}_1'} e^{\frac{\mathrm{B}_2'}{1-\xi}}\, d\xi$$

Further, we can use the transformed differential operator to **construct a scalar product** (symmetric bilinear form) (Green+) **Evaluation of scalar products is possible** via analytic continuation methods …

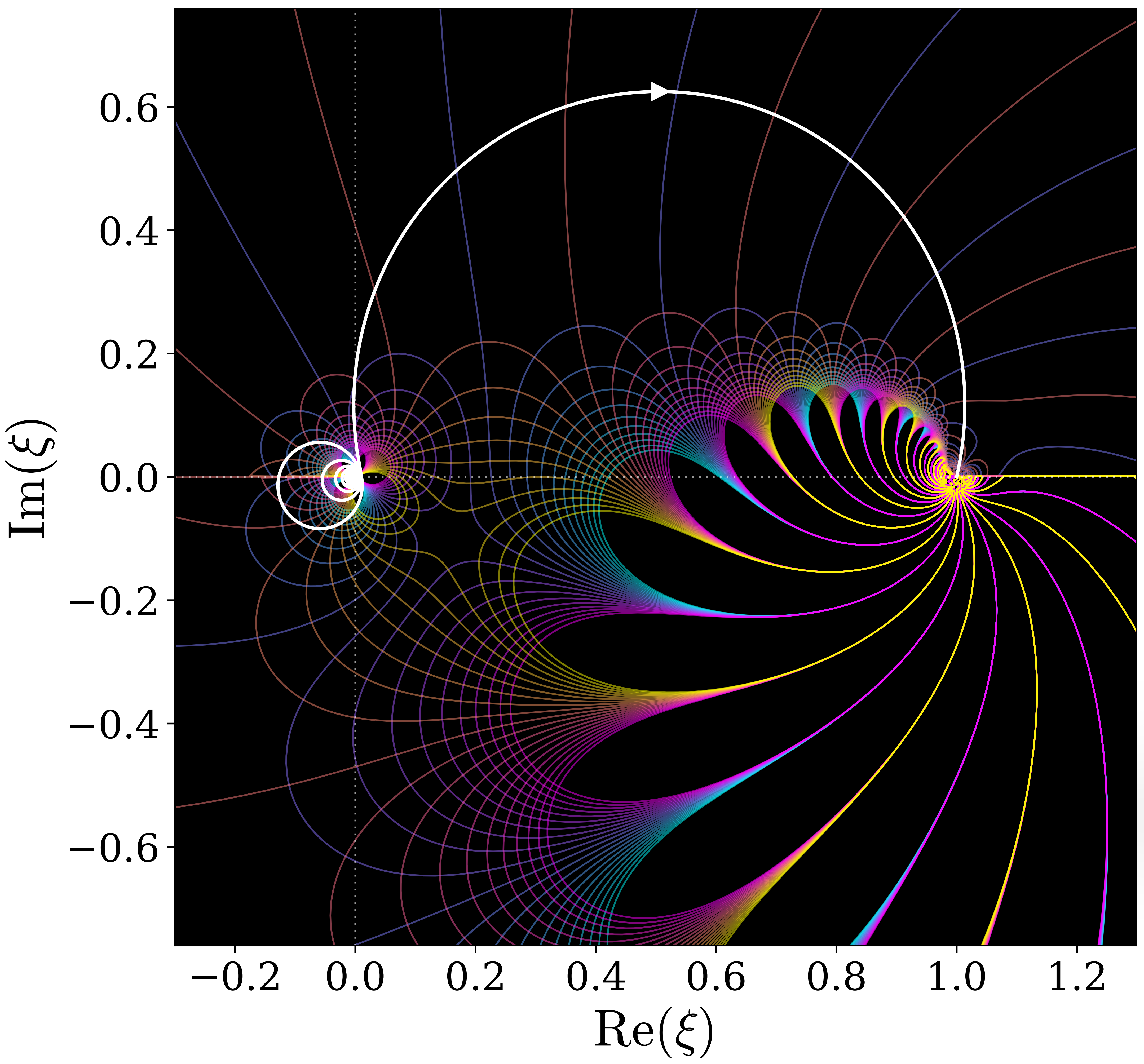
actional radial coordinates | **Study the scalar product** | Study the radial OD

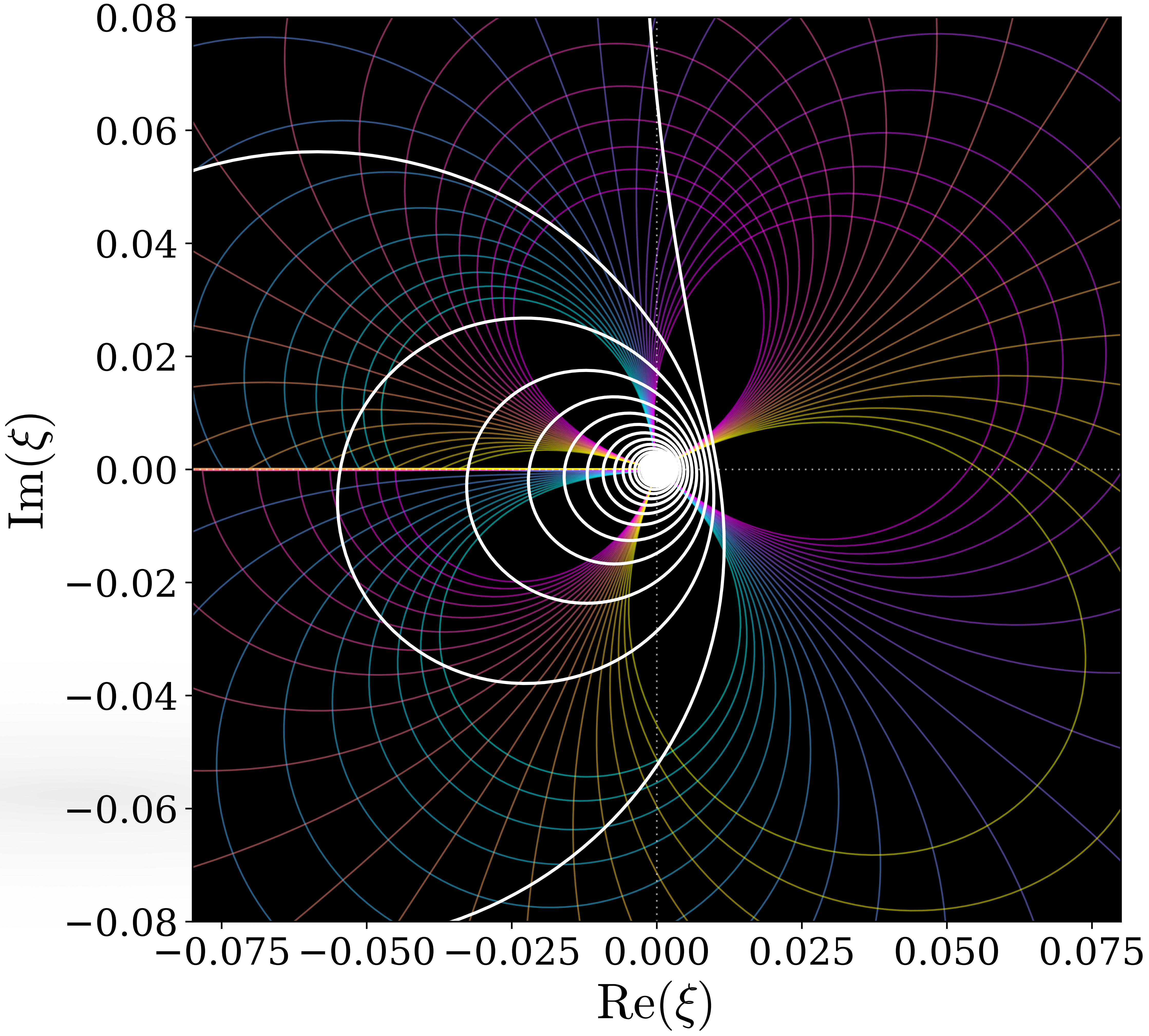
$$\langle \psi_j \,|\, \mathrm{W}_{r\theta} \,|\psi_k\rangle_{r\theta} \propto \delta_{jk}$$

$$\mathrm{W}_{r\theta} = -(\omega_j + \omega_k)\left(\frac{(a^2+r^2)^2}{\Delta(r)} - a^2\sin^2(\theta)\right) + 2is\left(\frac{r^2-a^2}{\Delta(r)} - ia\cos(\theta) - r\right) + \frac{4amr}{\Delta(r)}$$

Alternative and related approaches:

- Use conserved current arguments to construct a symmetric bilinear form.
- Apply Sturm-Liouville arguments (self-adjointness) to Teukolsky's wave operator





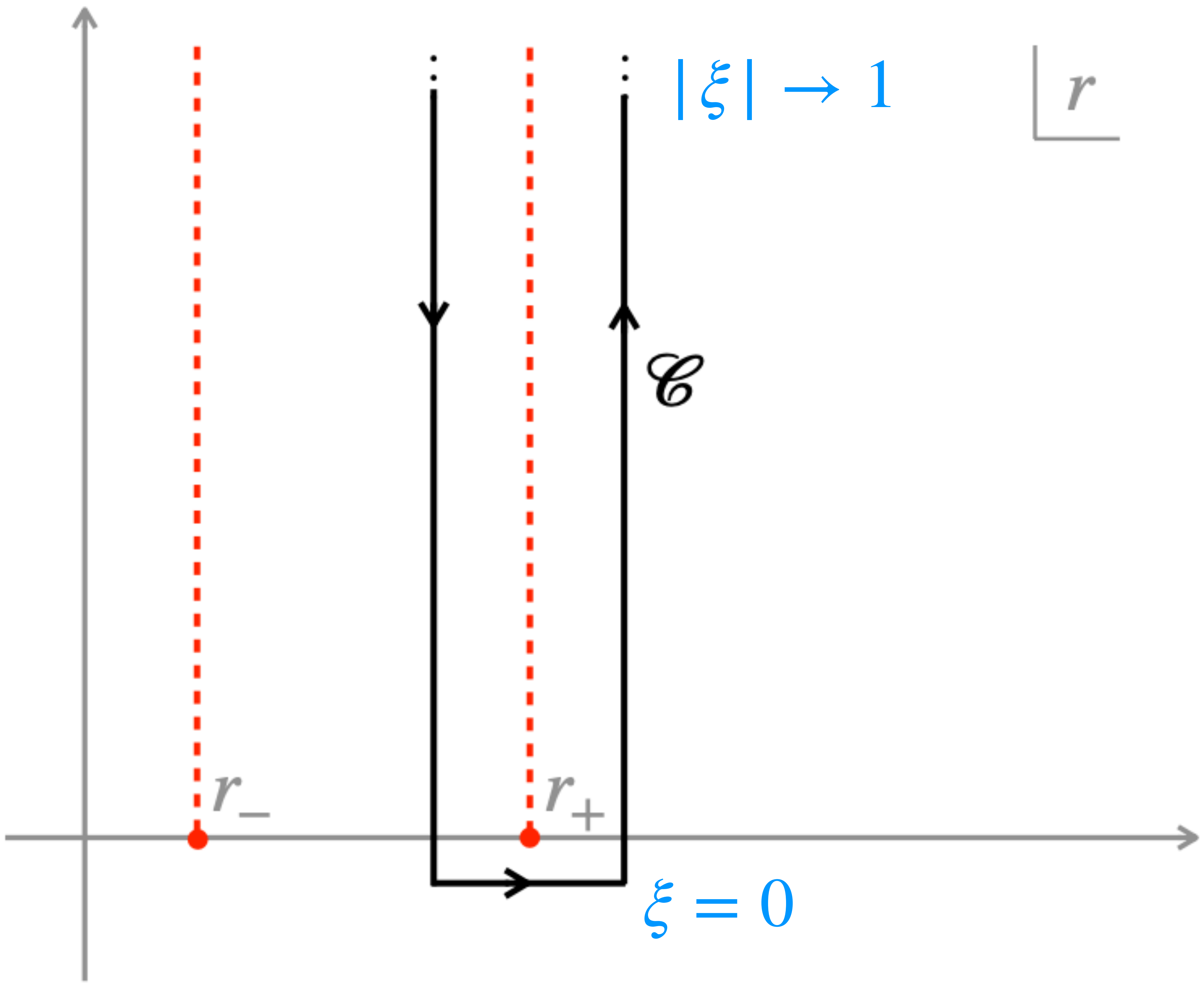
actional radial coordinates

Study the scalar product

Study the radial OD

Alternative (and equivalent) approaches:

- Pochhammer Contour around poles of integrand.
- Sum over confluent hypergeometric functions.
- Given a suitable orthonormal basis, use the standard dot product.



actional radial coordinates | **Study the scalar product** | Study the radial OD

$$\mathcal{L}_\xi = (\mathrm{C}_0 + \mathrm{C}_1(1-\xi)) + \mathcal{D}_\xi$$

$$\mathcal{D}_\xi = \left(\mathrm{C}_2 + \mathrm{C}_3(1-\xi) + \mathrm{C}_4(1-\xi)^2\right)\partial_\xi + \xi(\xi-1)^2\partial_\xi^2$$

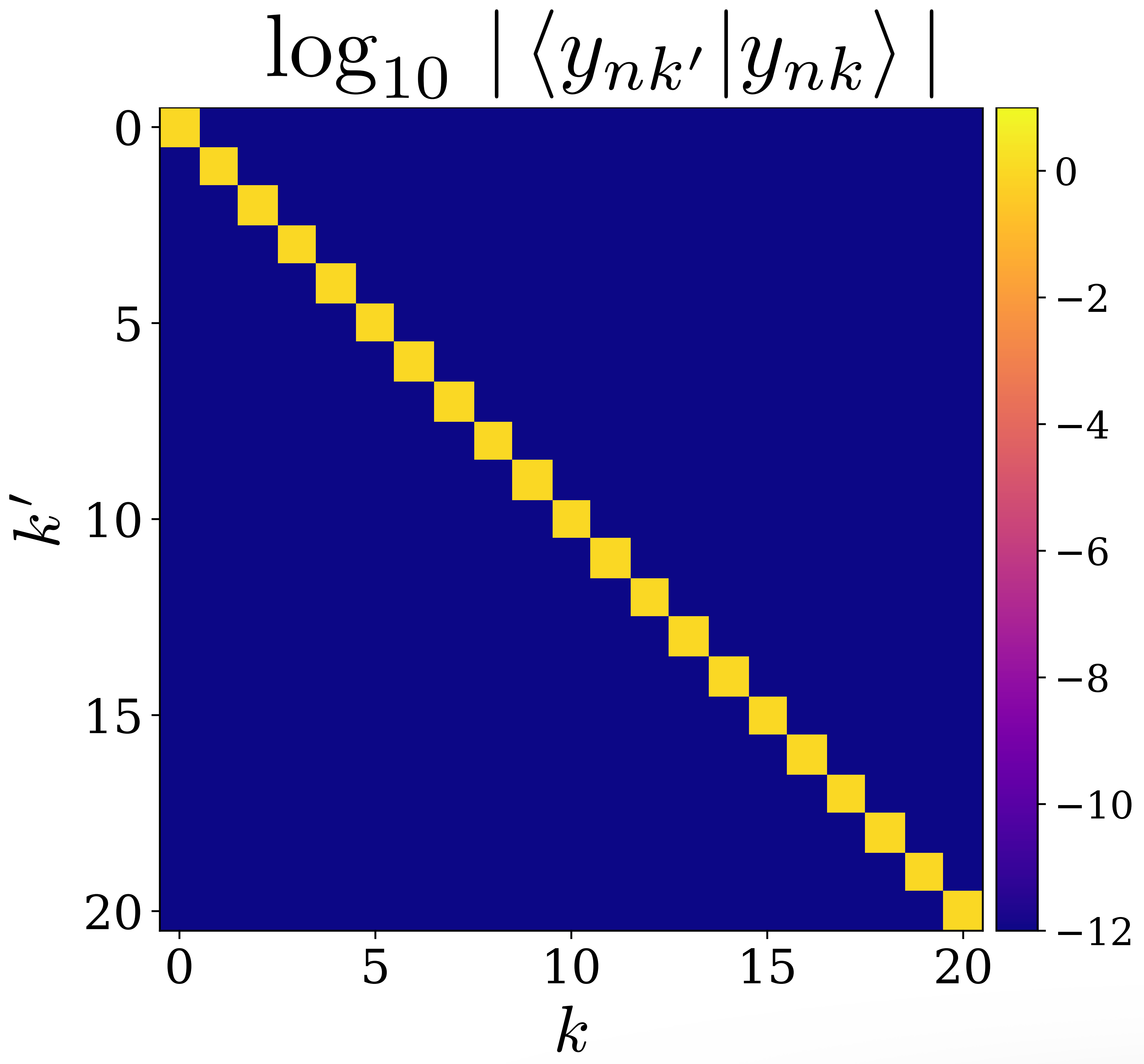
Key idea: If there exist a class of polynomials that are eigenfunctions of $\mathcal{D}_\xi$ then they would be extremely well positioned to simplify the determination of solutions to the physical problem

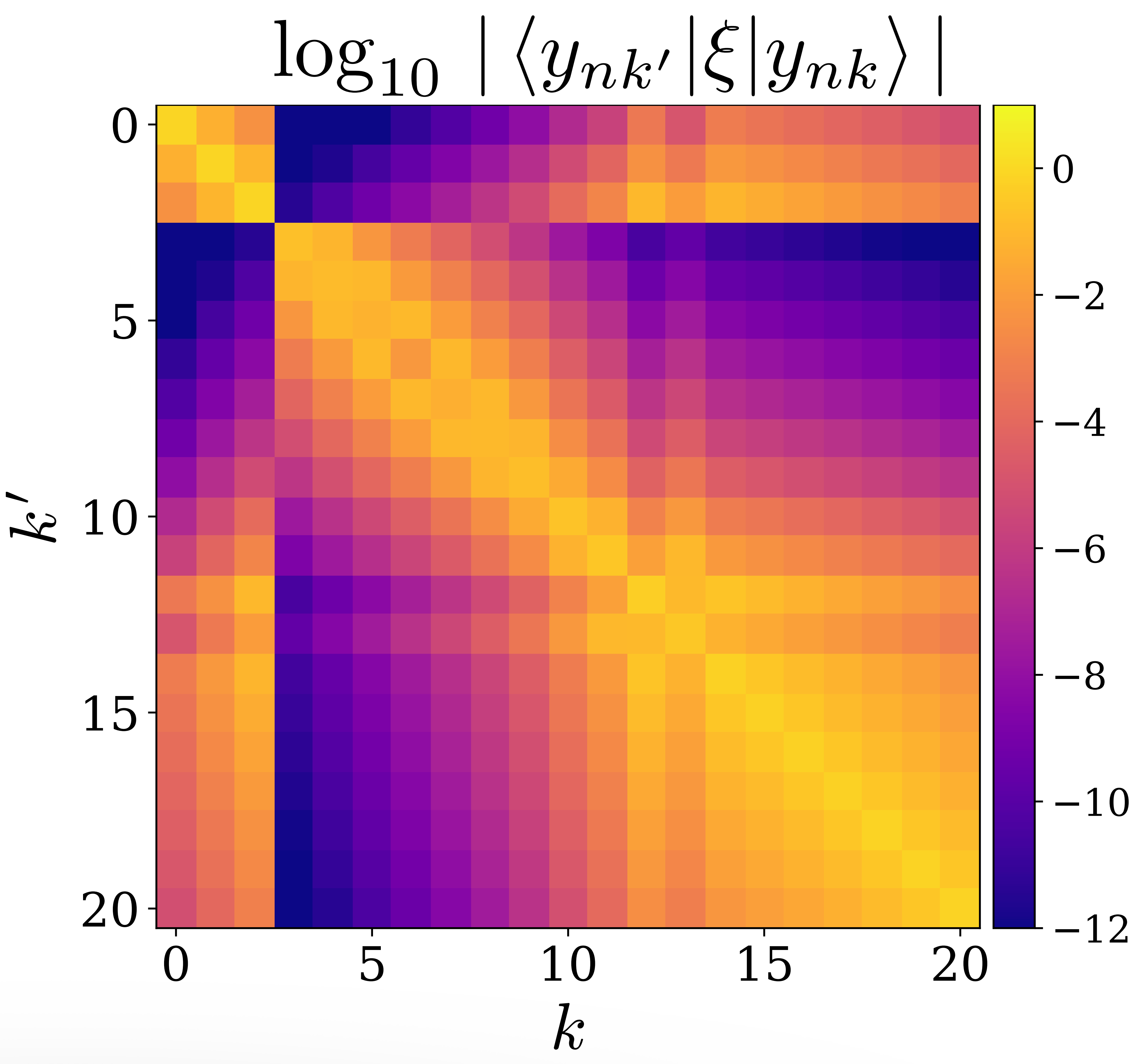
QNM boundary conditions | **Study the radial ODE** | Confluent Heun polyno

$$y_{nk} = \sum_{j}^{n} a_{jkn} \xi^j$$

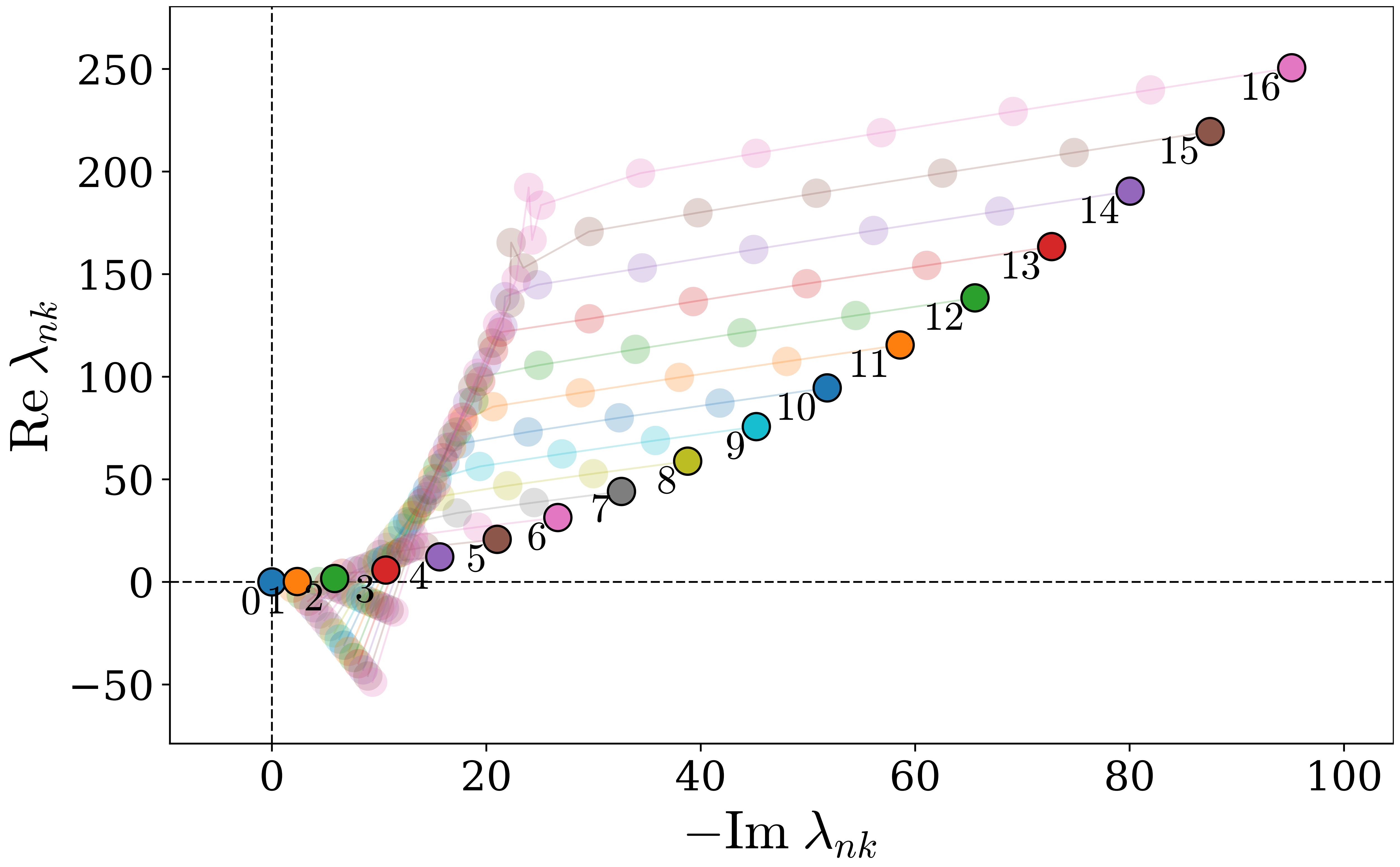
$$\mathcal{D}_\xi \, y_{nk}(\xi) = [\, \lambda_{nk} - n(n - 1 + \mathrm{C}_4)(1 - \xi) \,] \, y_{nk}(\xi)$$

It turns out that polynomial solutions are a two-parameter eigenvalue problem. **Confluent Heun polynomials** are the result …





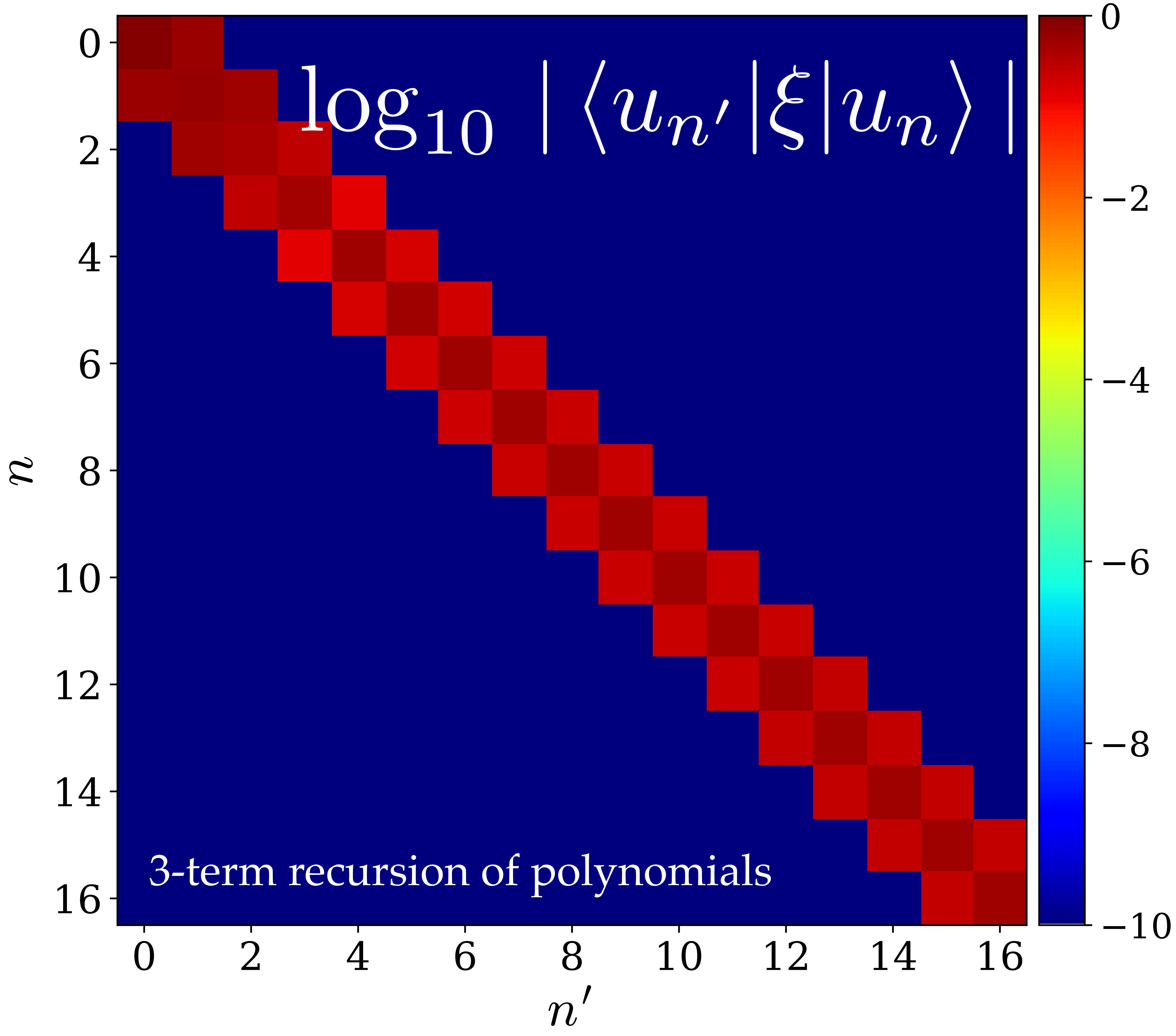
Using the scalar product, one can show orthogonally between polynomials of fixed order



- Dominant eigenvalues are typically distinct
- Parabolic dependence can be qualitatively understood

$$\lambda_{nk} = n\,(\,n + C_4 - 1\,)\,\frac{\langle y_{00}|\xi\,|y_{nk}\rangle}{\langle y_{00}\,|\,y_{nk}\rangle}$$

Since the polynomials are "non-classical", they are not sufficient to simplify the physical problem. **It happens that this is not a dead end ….**



"Canonical polynomials"

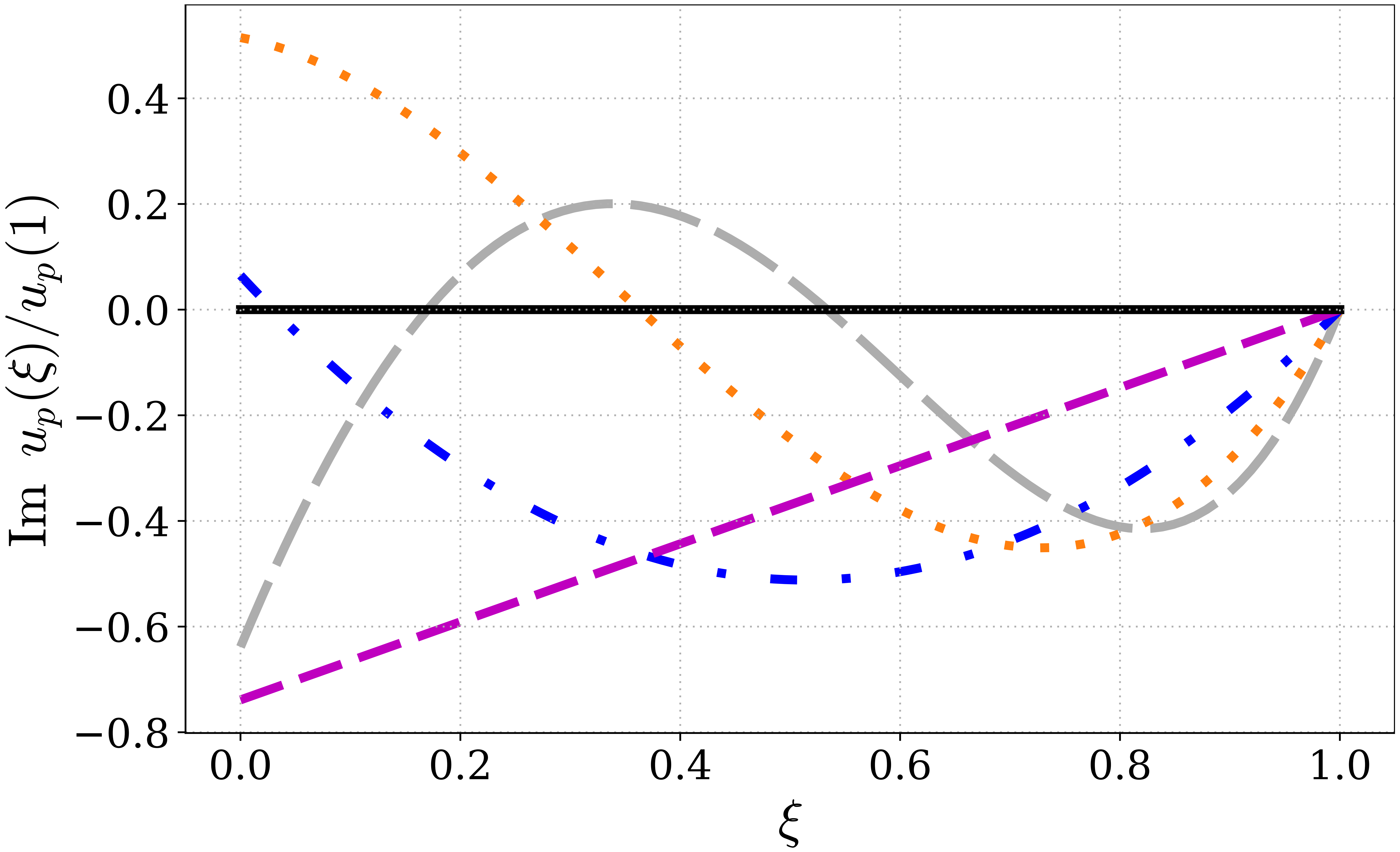
$$u_n(\xi) = \sum_{k=0}^{n} b_{nk}\, y_{nk}(\xi)$$

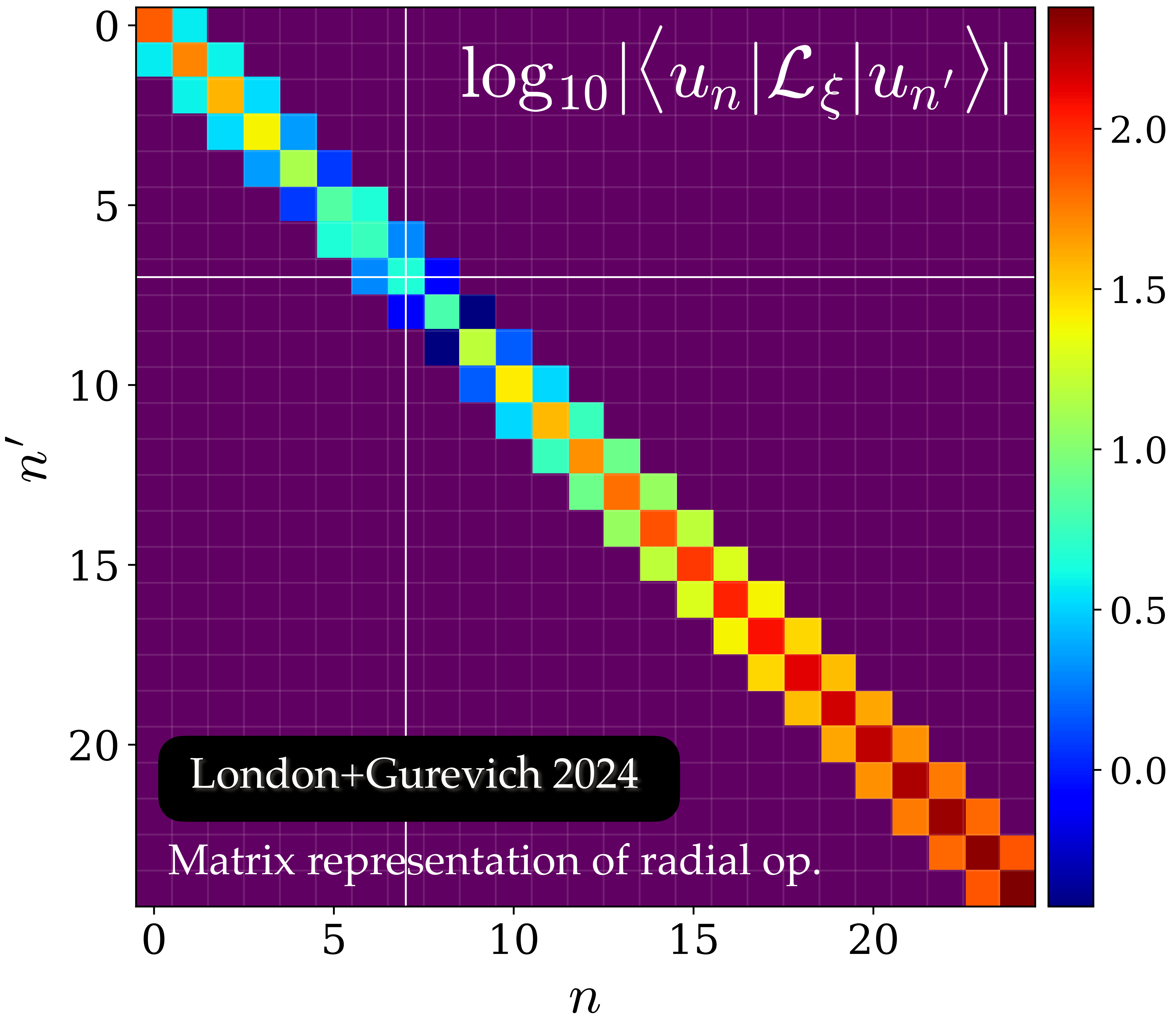
We have a multitude of options …

- "Canonical construction"
- Gram Schmidt …
- **All options result in equivalent polynomials**

ehavior of eigenvalues | Develop orthonormal sequence | Apply to the radial equa

Canonical polynomials (Pollaczek-Jacobi type)





$$\mathcal{L}_\xi\, f(\xi) \;=\; A\, f(\xi)$$

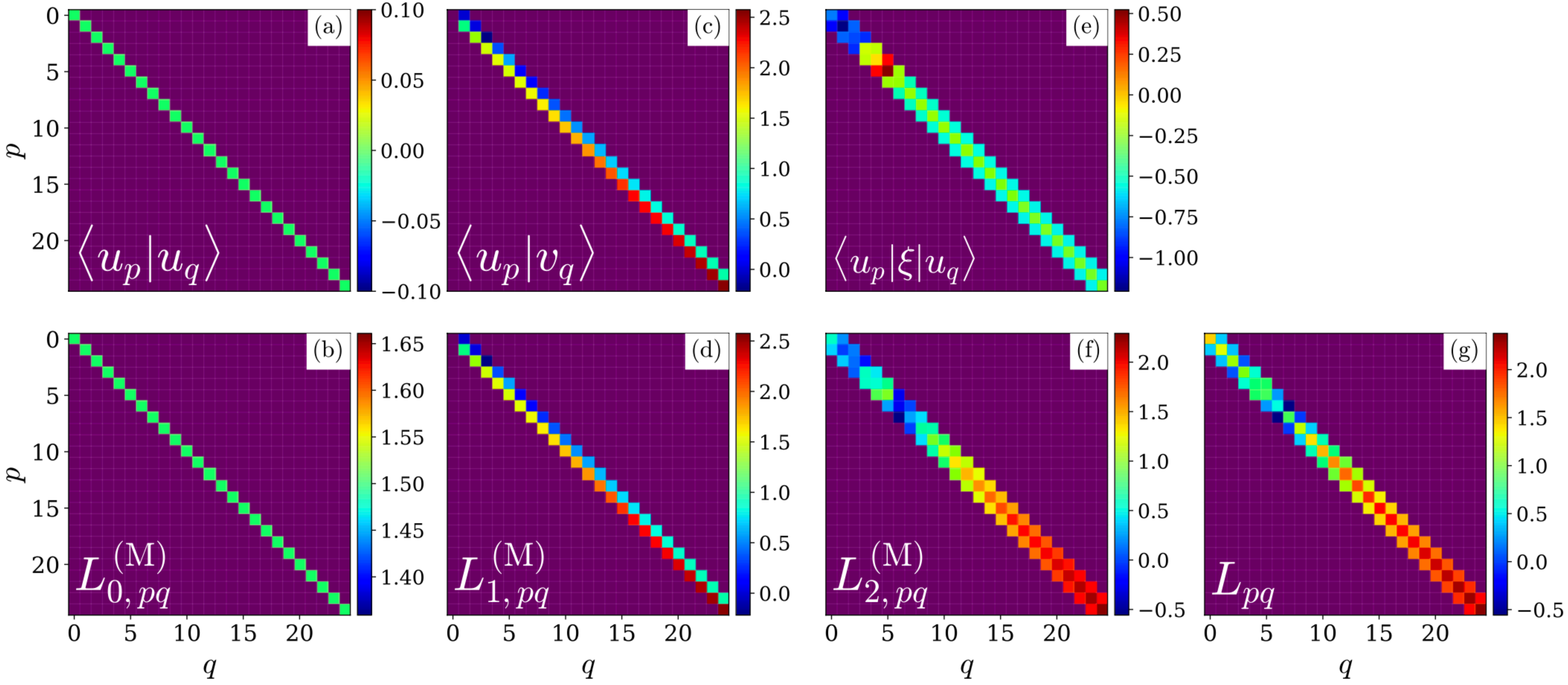
$$f(\xi) \;=\; \sum_{n=0}^{\infty} c_n\, u_n(\xi)$$

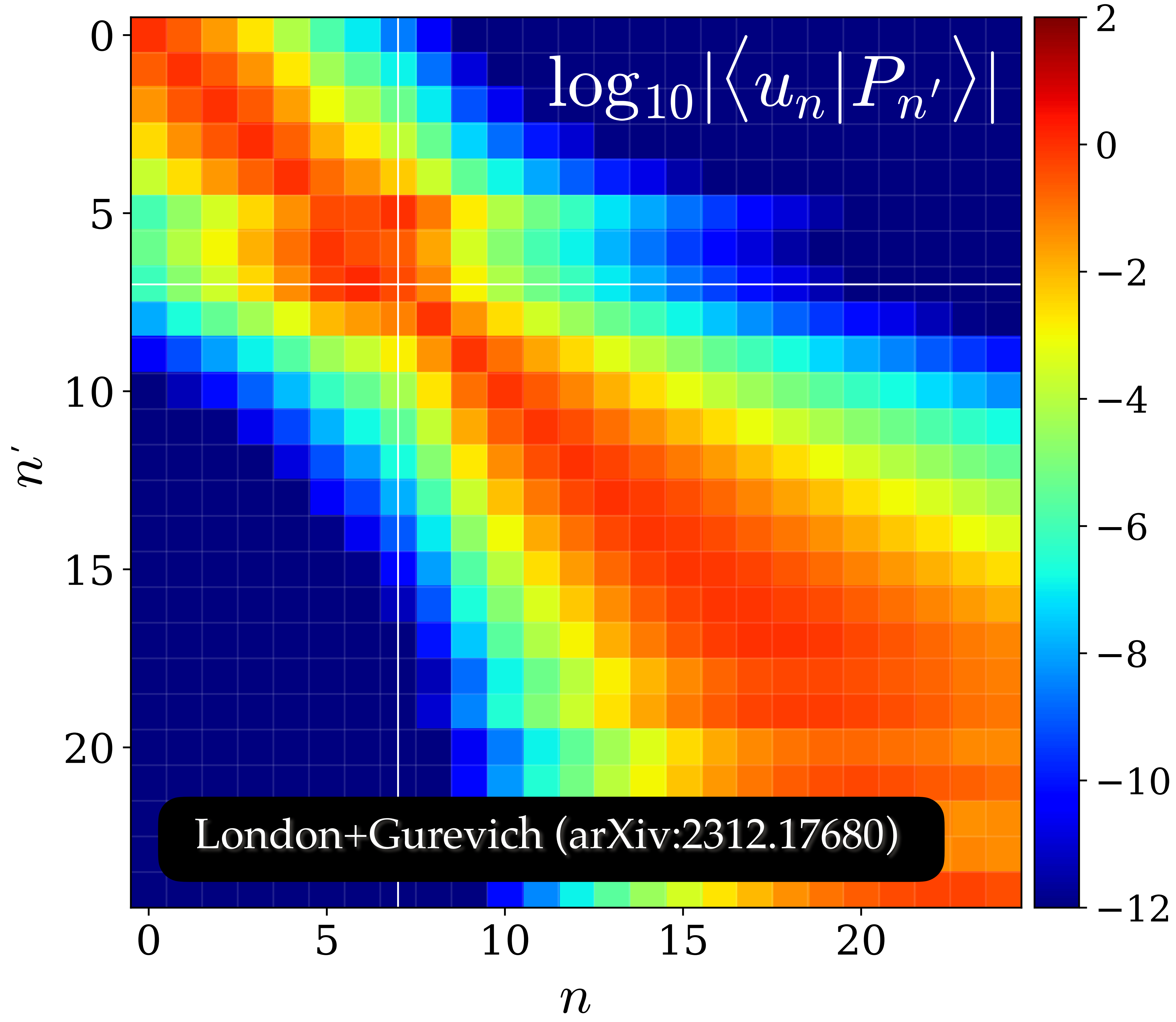
- **Matrix form of radial equation is symmetric**
- **Exactly tridiagonal due to properties properties of our "canonical polynomials"**

elop orthogonal sequence

Apply to the radial equation

Tridiagonalization (step-by-step)





$$\mathcal{L}_\xi\, f(\xi) \;=\; A\, f(\xi)$$

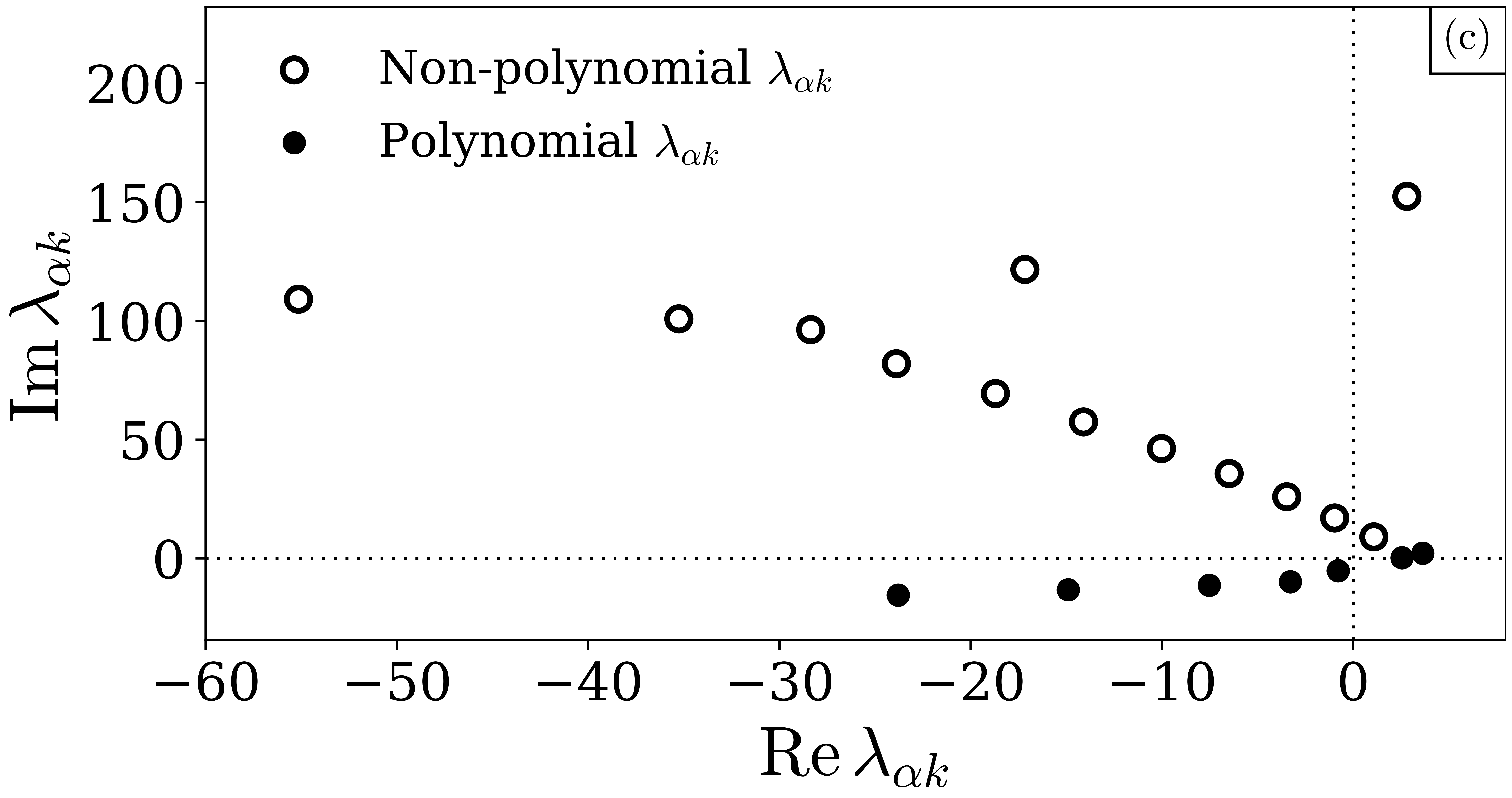
$$f(\xi) \;=\; \sum_{n=0}^{\infty} c_n\, u_n(\xi)$$

- $c_n \;=\; \langle u_n|P_n\rangle$
- Results are for **fixed frequency parameter**
- Eigenvectors are orthogonal and "complete" (*caution*)

elop orthogonal sequence

Apply to the radial equation

Analogy with quasi-bound states



Example Radial Eigenvalues

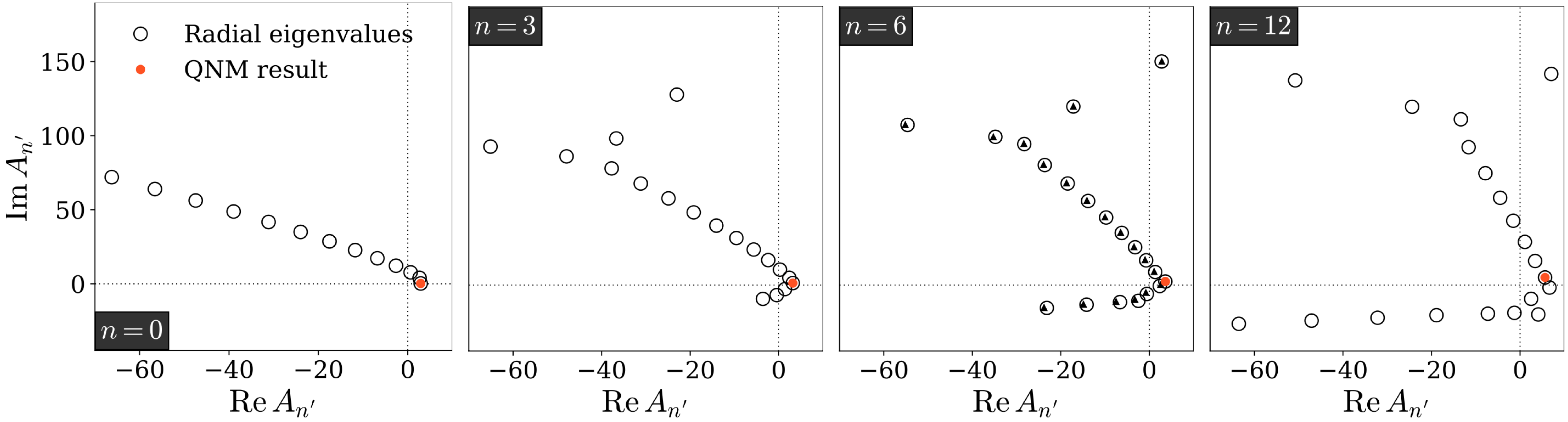
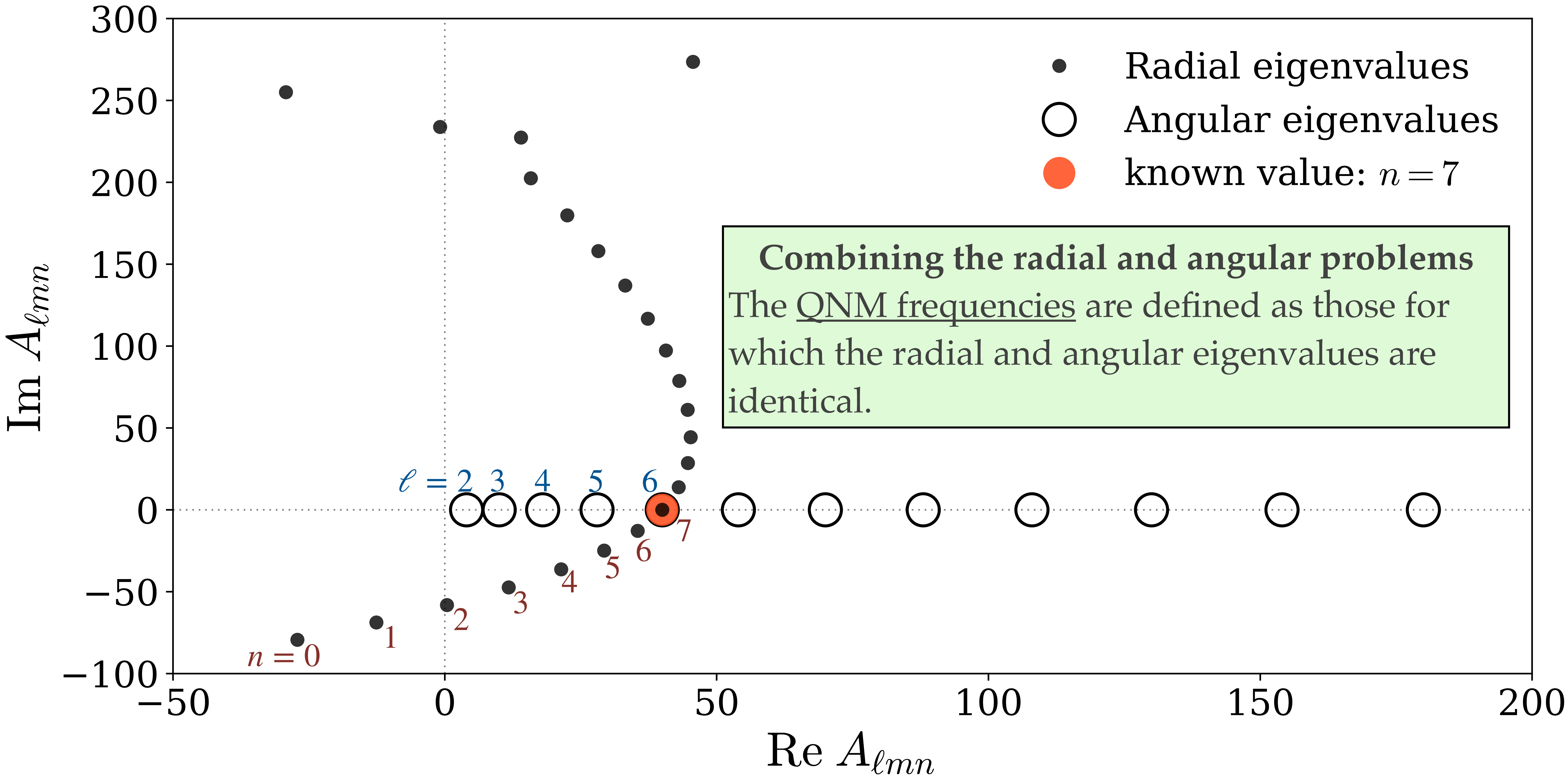


FIG. 5. Radial eigenvalue distributions for select quasinormal mode cases, all with BH dimensionless spin $a/M = 0.7$, and angular indices $(\ell, m) = (2, 2)$. In all panels, radial eigenvalues, $A_{n'}$, are noted with open circles, and the known QNM eigenvalue, $A_{\ell mn}$, is shown as a red filled circle (c.f. Fig. 1). Left to right: Radial eigenvalues for the respective $n \in \{0, 3, 6, 12\}$ QNM overtones. Respective absolute differences between known QNM eigenvalues (Refs. [108, 109]) and those computed here (i.e. $|A_{\ell mn} - A_n|$) are $\{2.72 \times 10^{-9}, 4.72 \times 10^{-9}, 1.24 \times 10^{-9}, 6.28 \times 10^{-9}\}$. Center right ($n = 6$): Eigenvalues for the closest confluent Heun polynomial problem are shown as black triangles. See panel (c) of Fig. 4 for comparison.



Combining the radial and angular problems

The QNM frequencies are defined as those for which the radial and angular eigenvalues are identical.

elop orthogonal sequence

Apply to the radial equation

Basic validation of spectral solutions

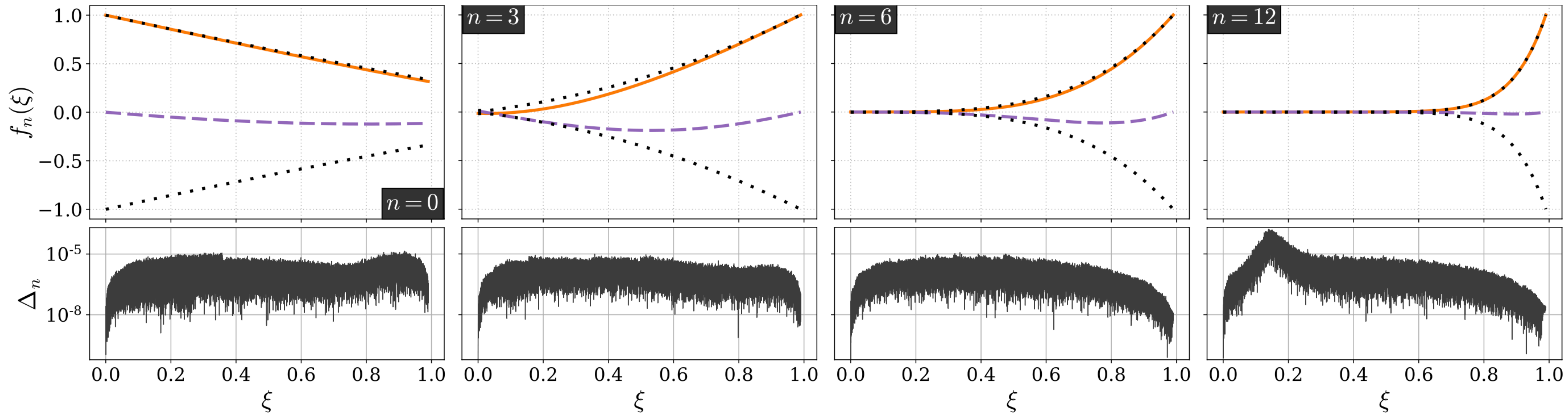
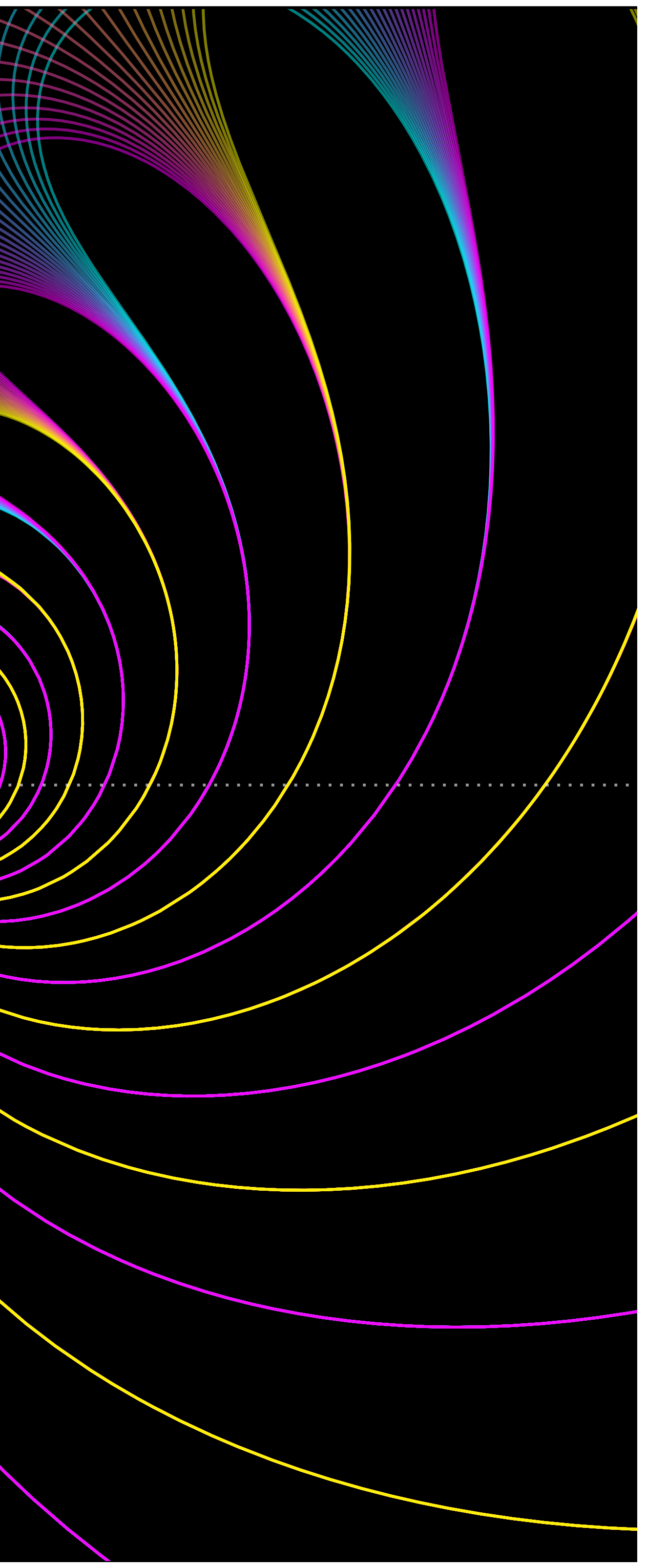
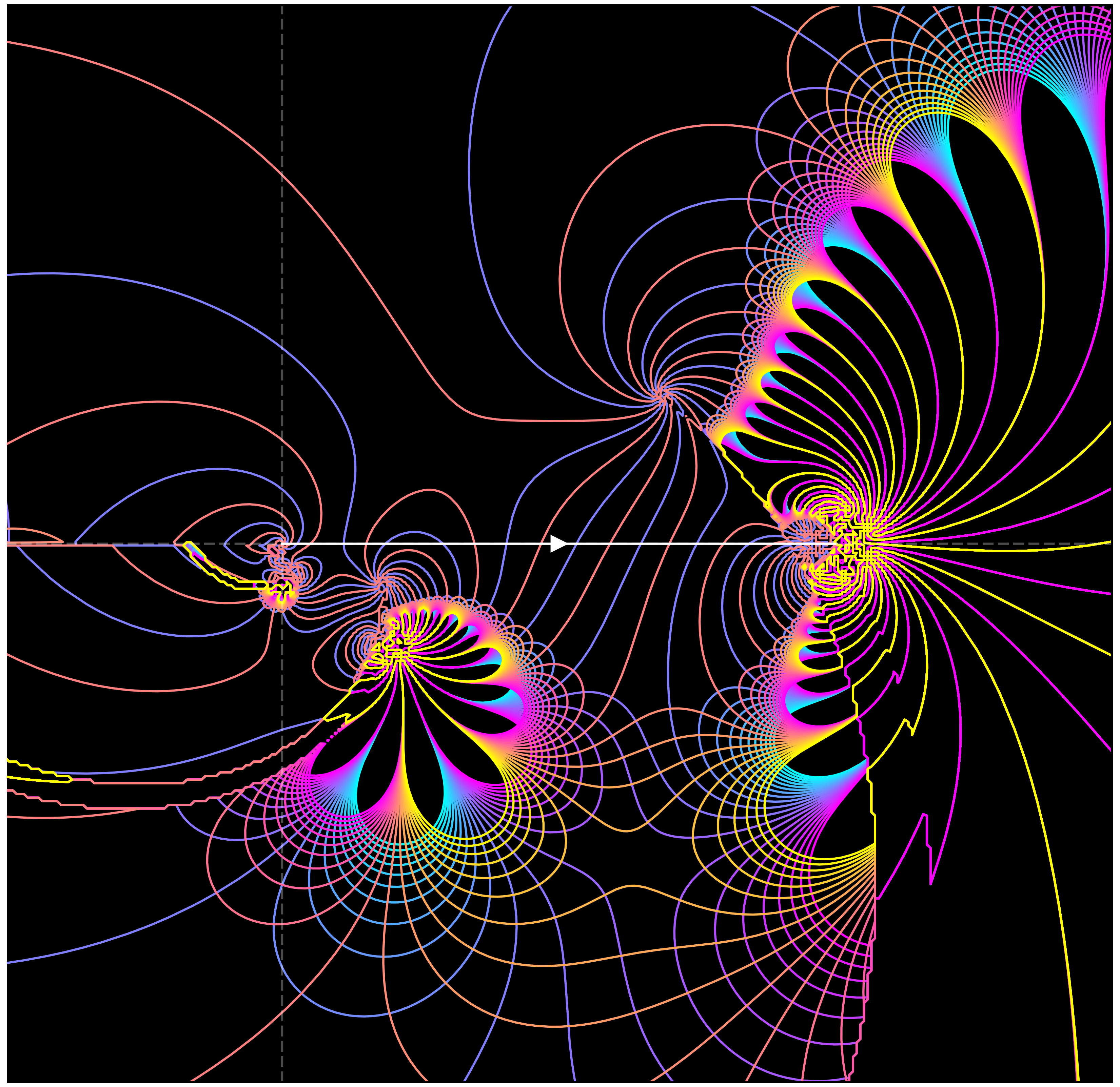
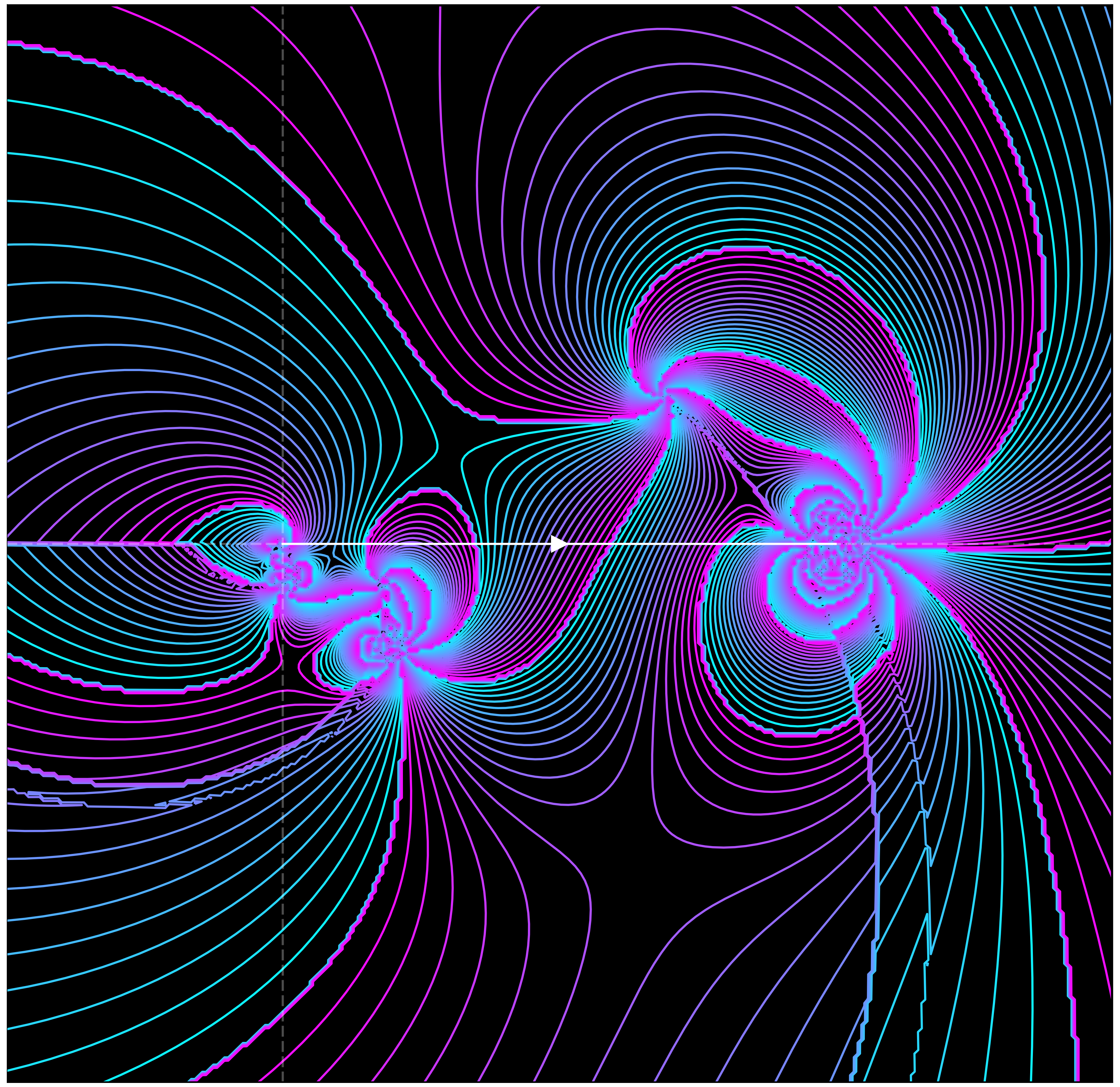
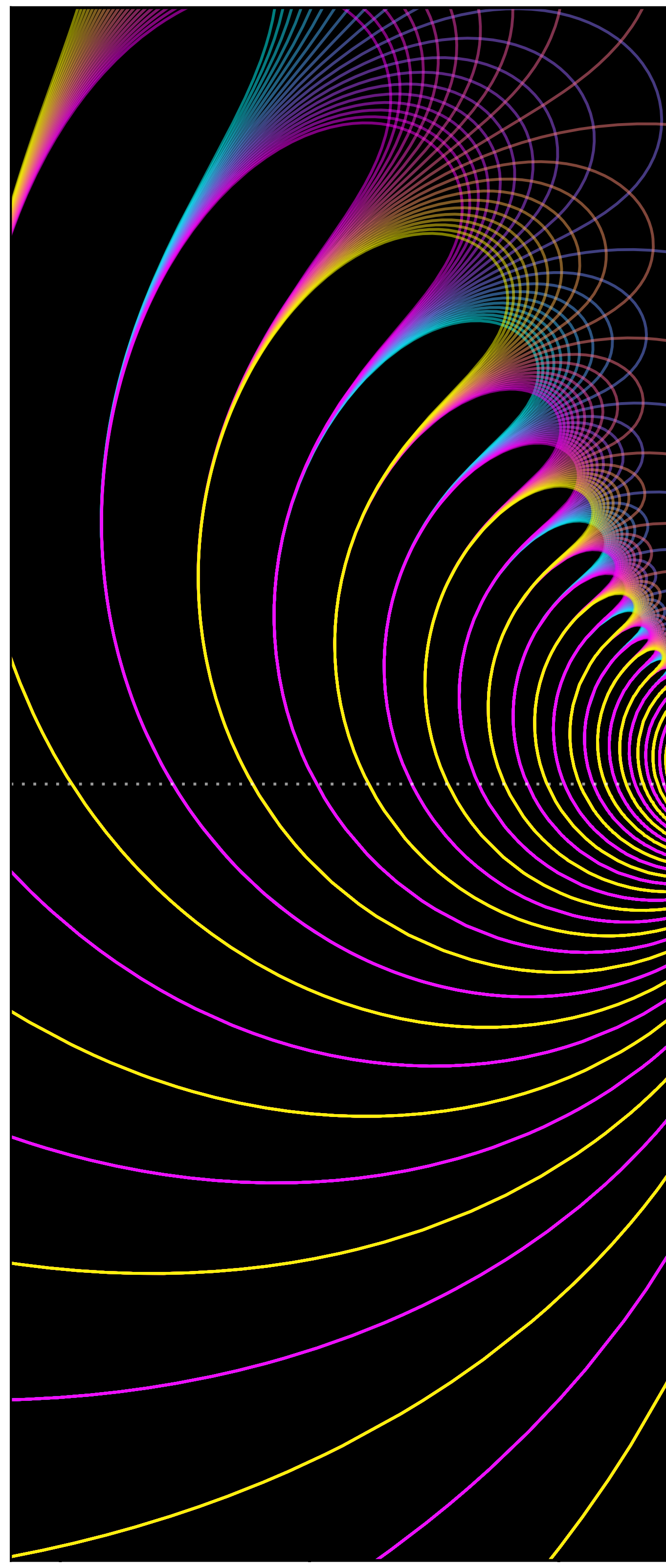


FIG. 6. Radial functions, $f_n(\xi)$, and related floating point error estimates, Δ_n, for the same cases shown in Fig. 5. Top row: The real part of each radial function, $\mathrm{Re}\, f_n(\xi)$, is denoted with a solid orange line. The imaginary part, $\mathrm{Im}\, f_n(\xi)$, is denoted with a dashed purple line, and the function's positive and negative magnitude, $\pm|f_n(\xi)|$ are denoted with dotted black lines. Bottom row: errors for each radial function, Δ_n (Eq. 98). See Sec. VII for related discussion.

Recap and conclusions

- Black hole quasi-normal modes (QNMs) are eigenfunctions of Einstein's equations linearized around a black hole solution. They are closely related to known and still somewhat lesser known special functions (e.g. confluent Heun functions).
- QNMs support GW signal models in LIGO (+future) and they remain a very active area of research within classical GR and beyond
- However, there is not yet consensus on the way forward — particularly regarding orthogonality, (over)completeness, and applications. Implications for beyond GR are of future interest …

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