THE CENTER OF GRAVITY

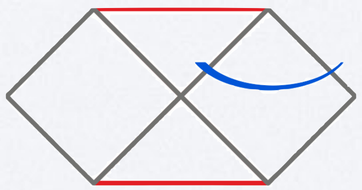
VILLUM FONDEN

KØBENHAVNS UNIVERSITET

Danmarks Grundforskningsfond
Danish National Research Foundation

THE HYPERBOLOIDAL FRAMEWORK FOR WAVE EQUATIONS

Rodrigo Panosso Macedo



https://hyperboloid.al

OUTLINE

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Phil. Trans. Roy. Soc. Lond. A 382 2267 (2024)
ArXiv: 2307.1573
Front. Phys. 12:1497601 (2025)
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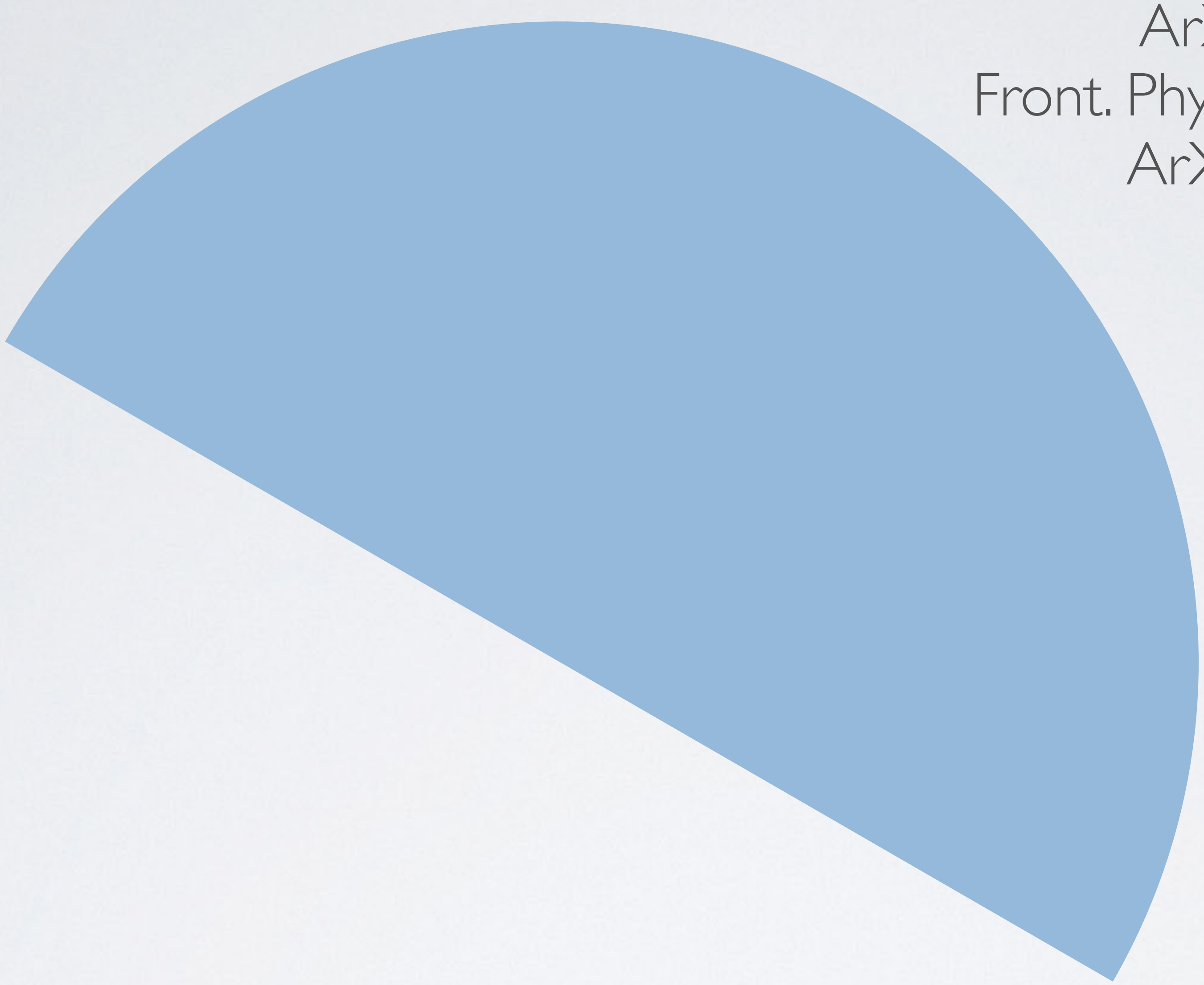

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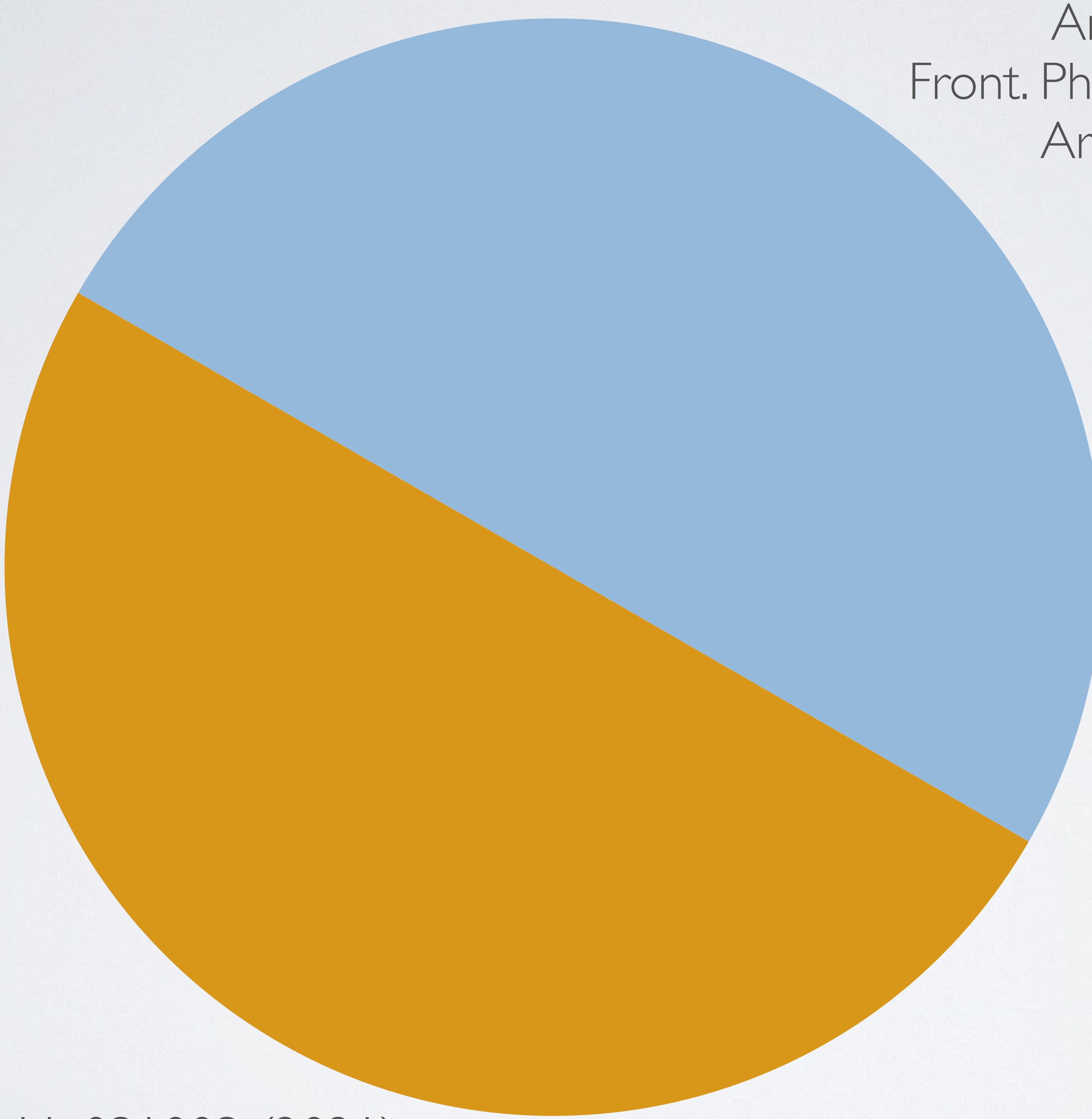

Phys. Rev. X 11, 031003 (2021)
ArXiv 2004.06434

QUASINORMAL MODES

- **Schrödinger-like Equation**

$$\left[\partial^2_{xx} - \mathcal{P}\right]\psi = -\omega^2\psi$$

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Dynamical Stability: Time evolution remains finite and decay in time

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- **Wave equation**

$$-\Psi_{,tt} + \Psi_{,xx} - P\Psi = 0$$

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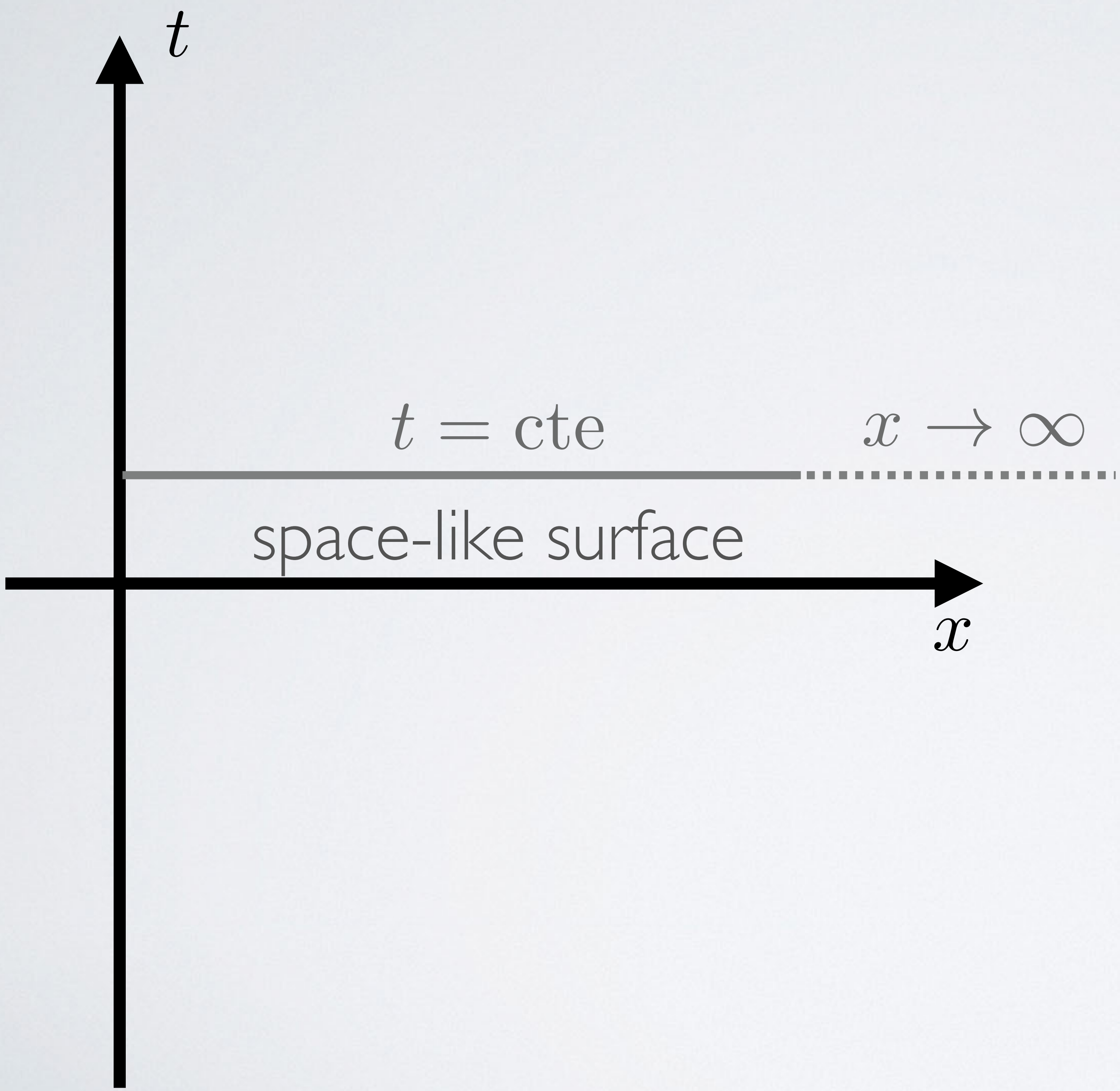
Wave phenomena relies upon a geometrical structure on space+time.

Limits on spatial coordinates are taken along surfaces of constant time.

What's the meaning of time? Which time coordinate?

COORDINATES



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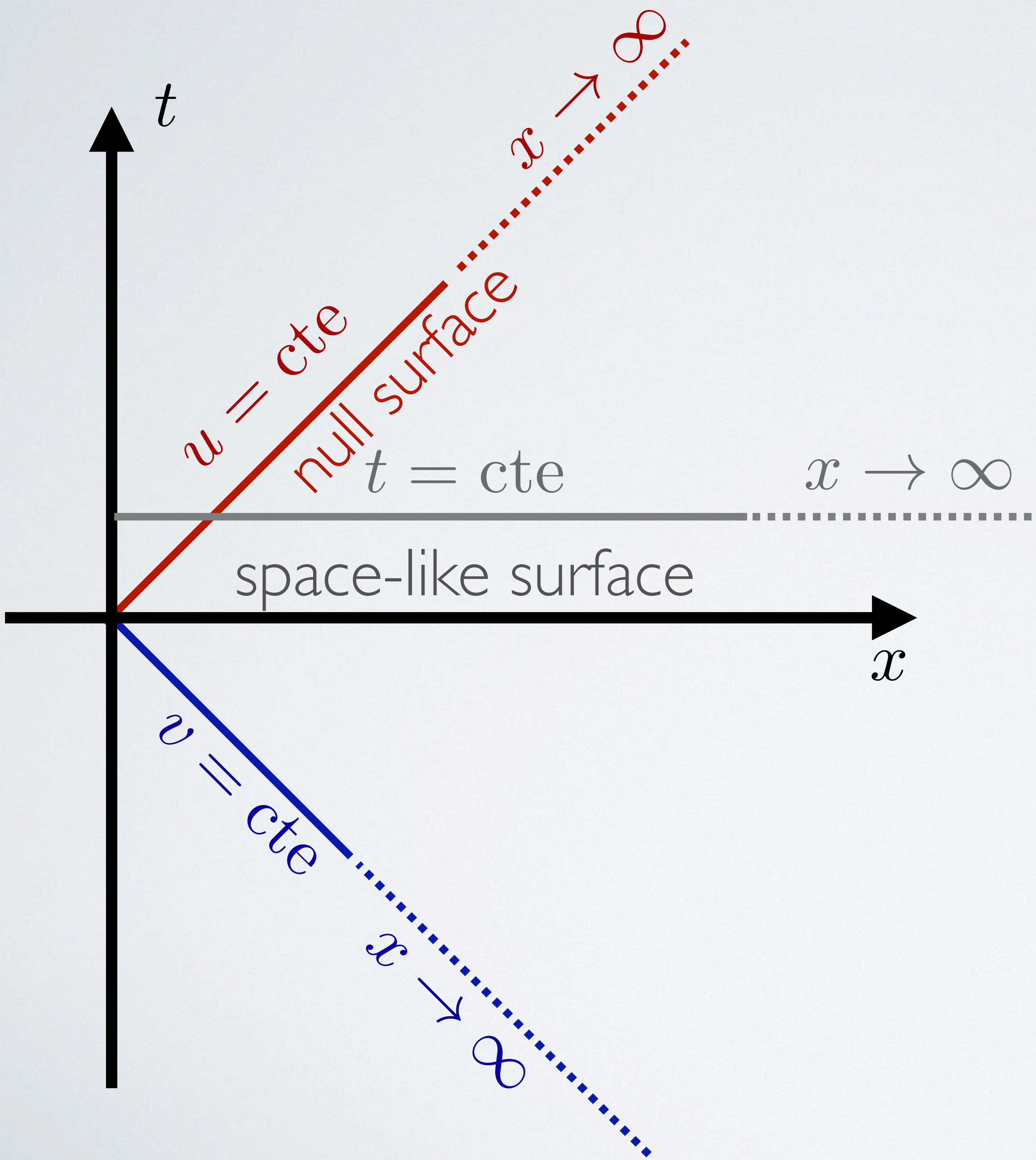
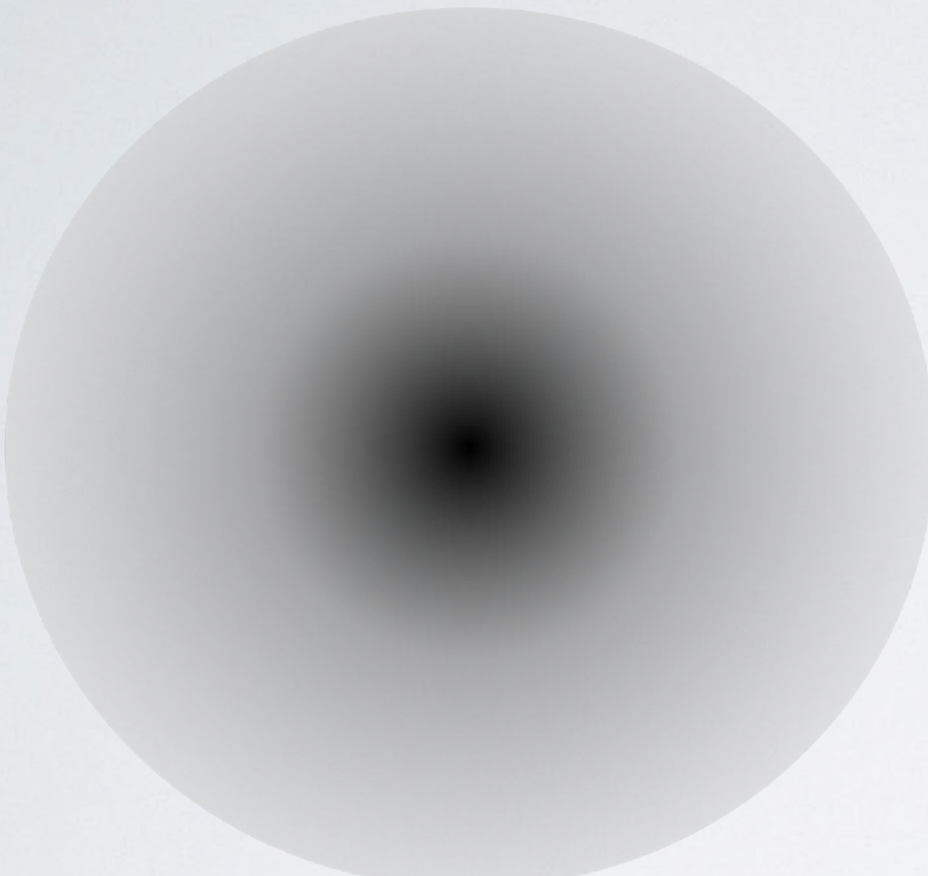


CONFORMAL COMPACTIFICATION

geometrical structure on space+time: Lorentzian Manifold



Physical manifold $(\mathcal{M}, g)$

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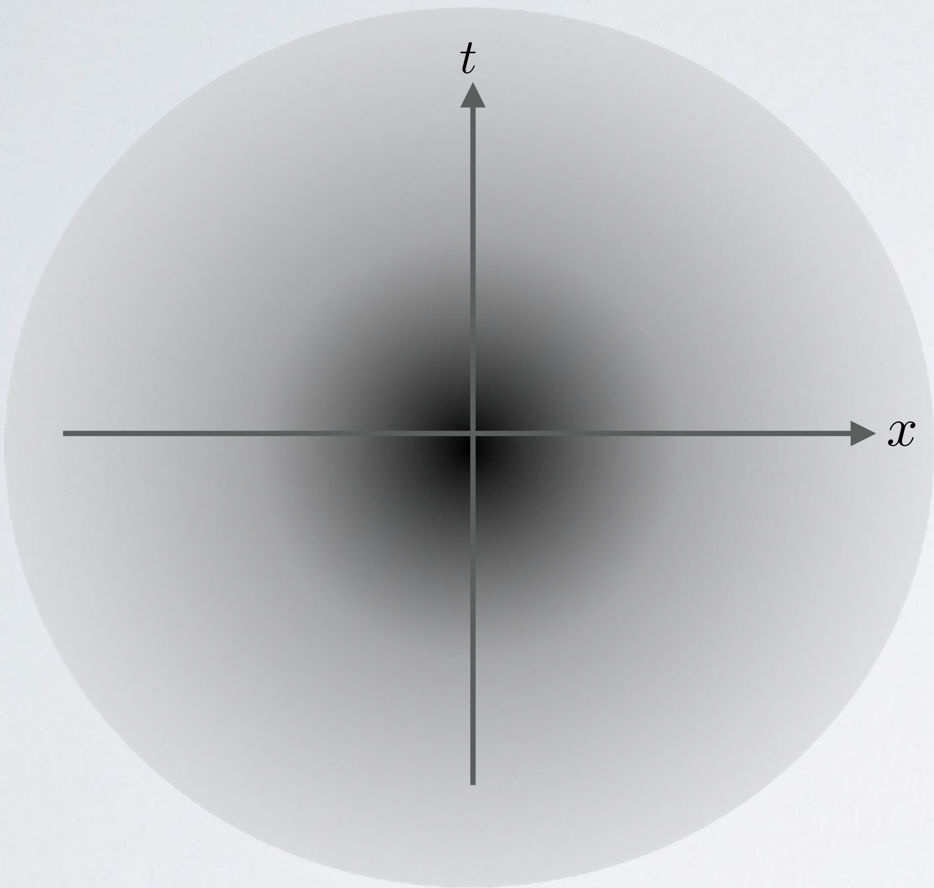

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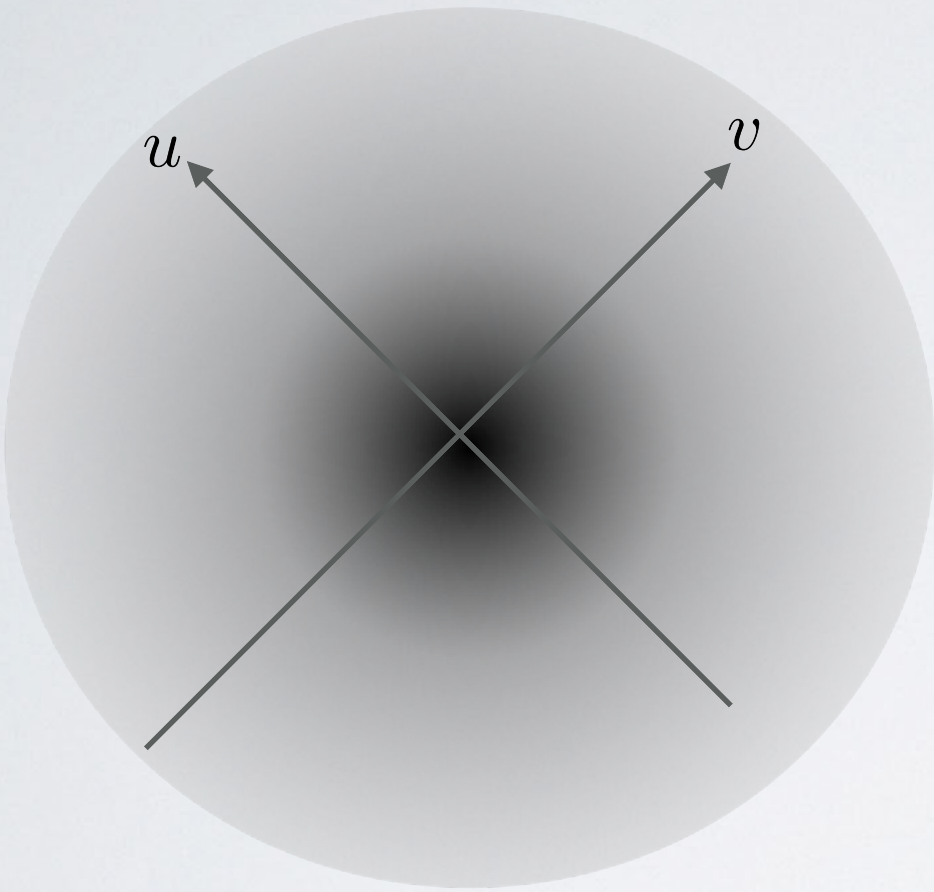

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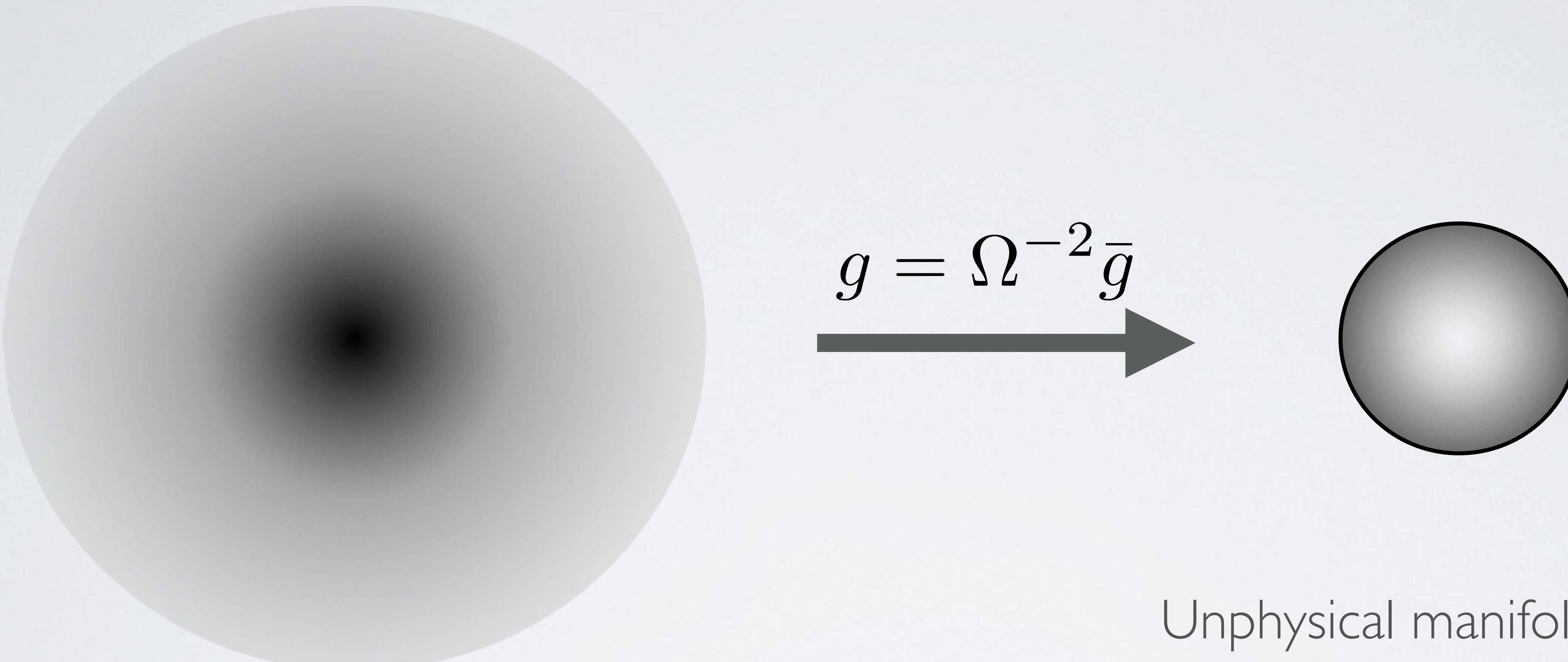

Unphysical manifold $(\bar{\mathcal{M}}, \bar{g})$

Boundary: $\partial\bar{M}$ $(\Omega = 0)$

Physical manifold $(\mathcal{M}, g)$

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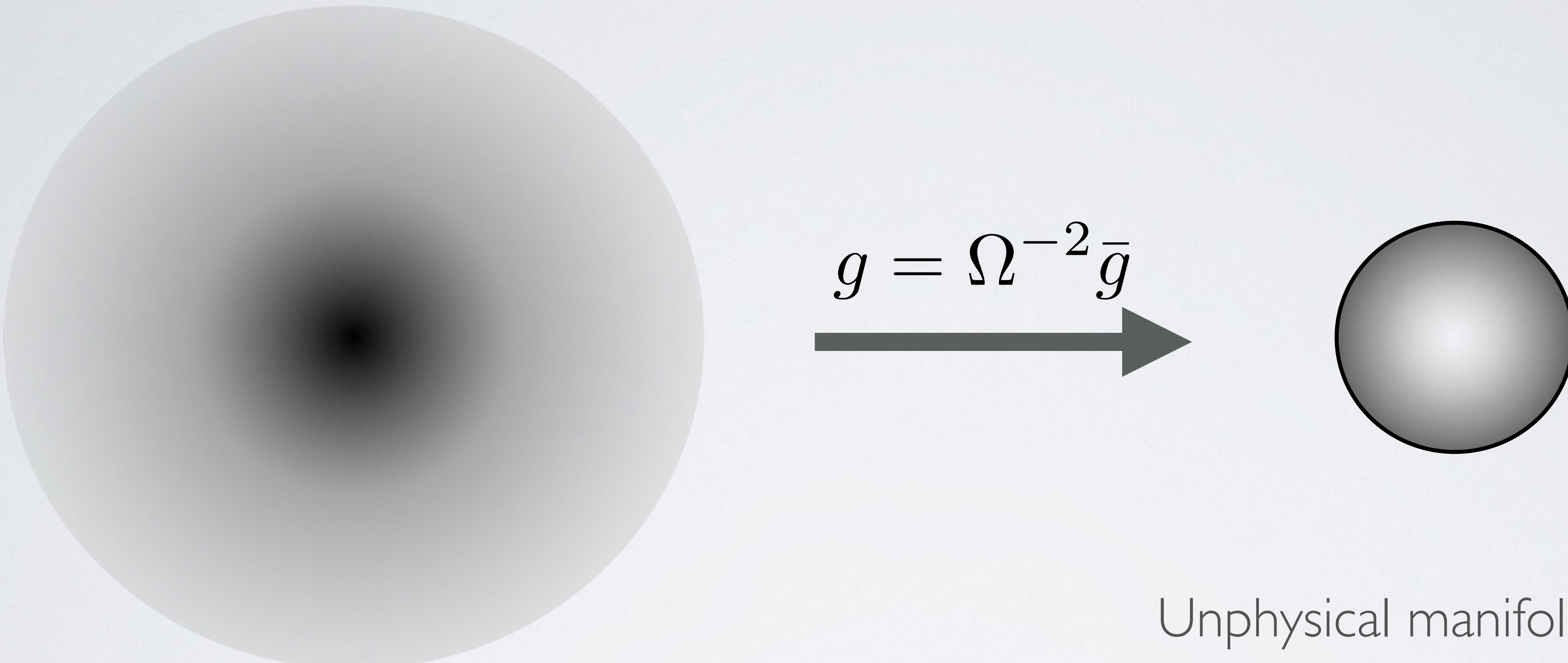

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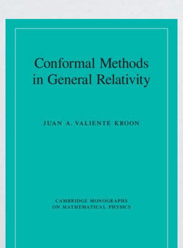
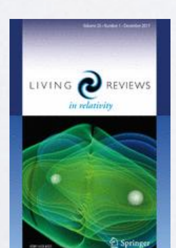
Conformal Methods in General Relativity

Juan A. Valiente Kroon



Conformal Infinity

Jörg Frauendiener



The conformal structure of space-times: Geometry, Analysis and Numerics

J. Frauendiener, H. Friedrich

At the interface of asymptotic, conformal methods and analysis in general relativity

Juan A. Valiente Kroon, G. Taujanskas

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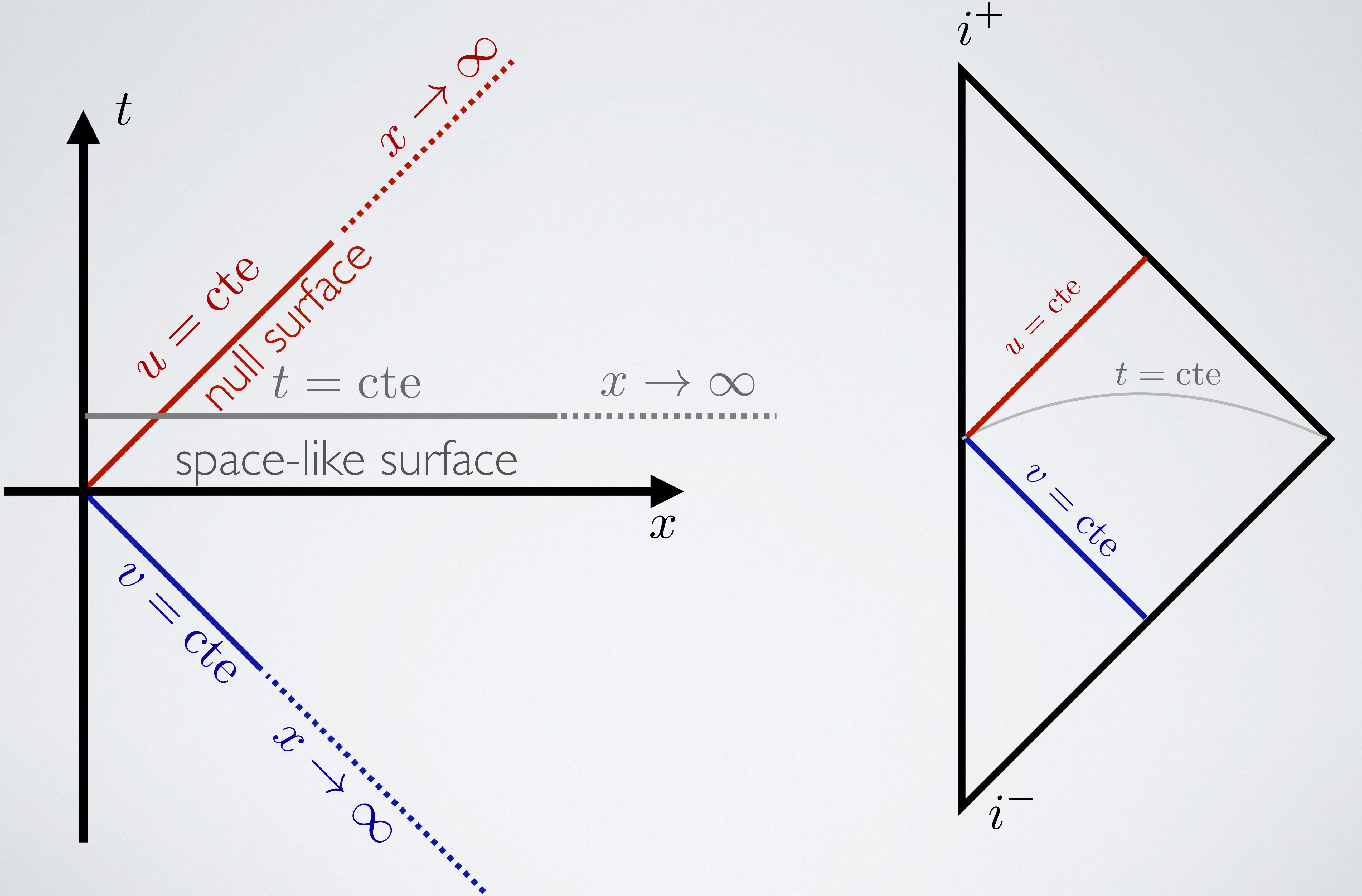

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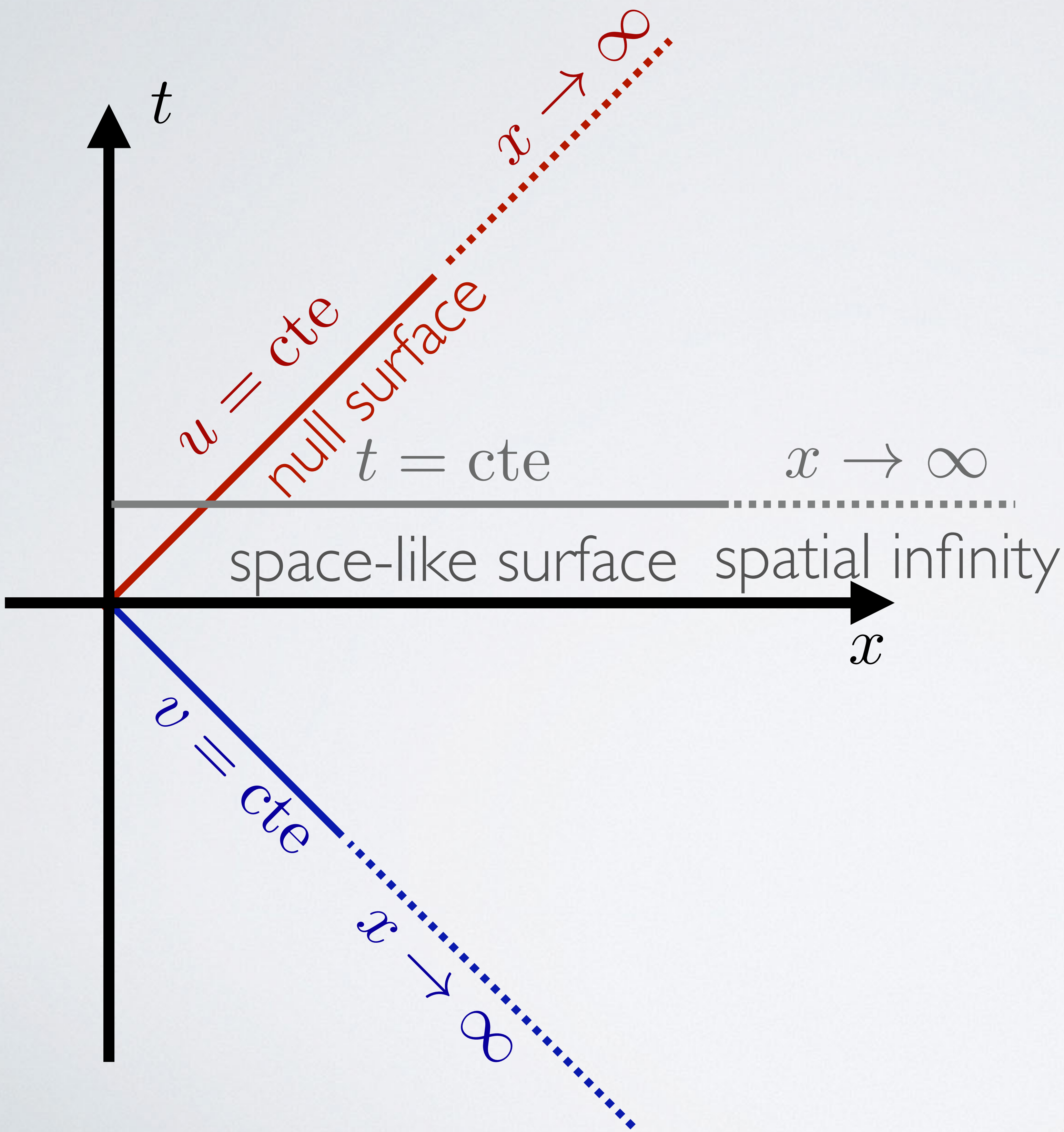

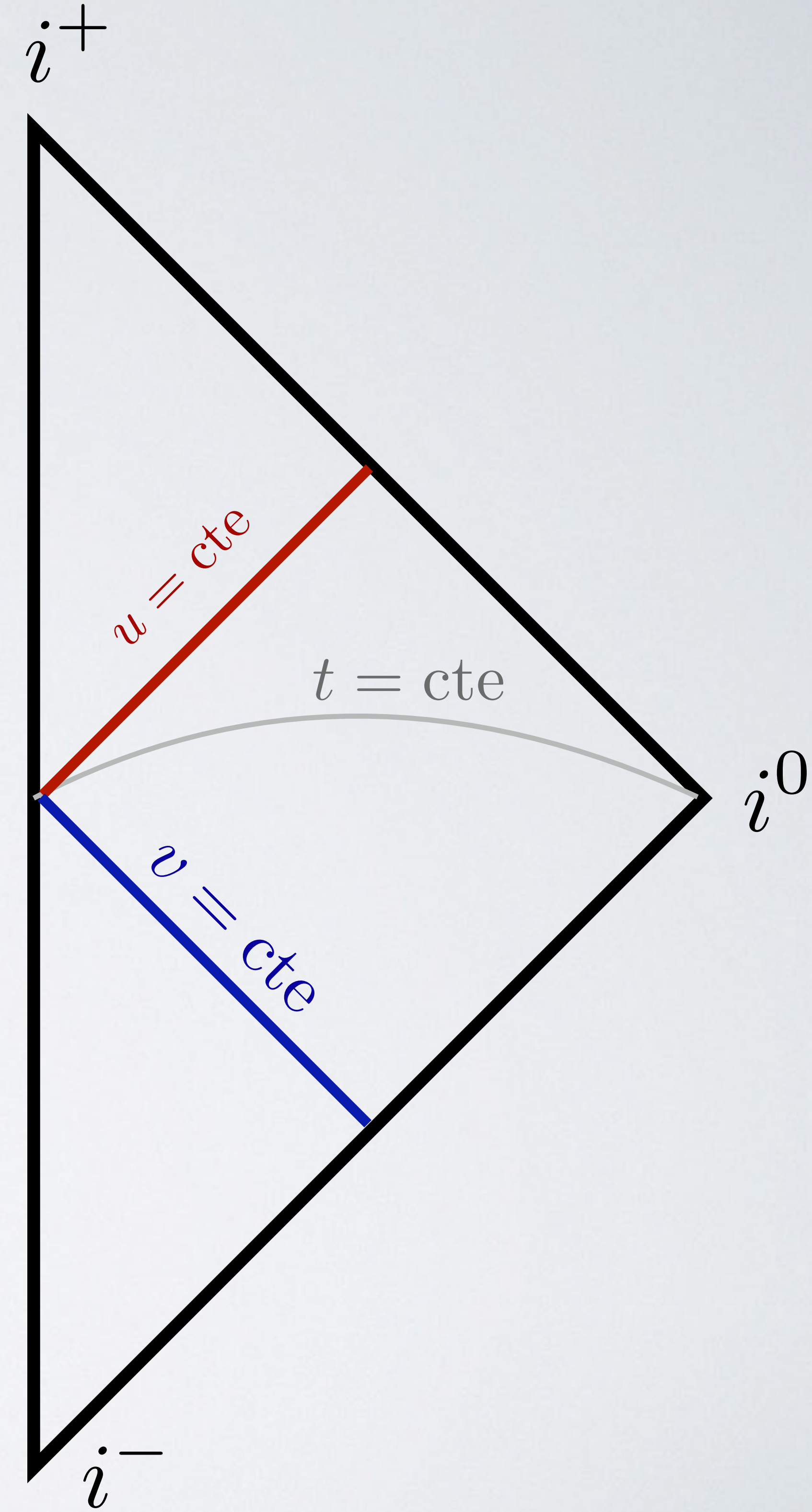

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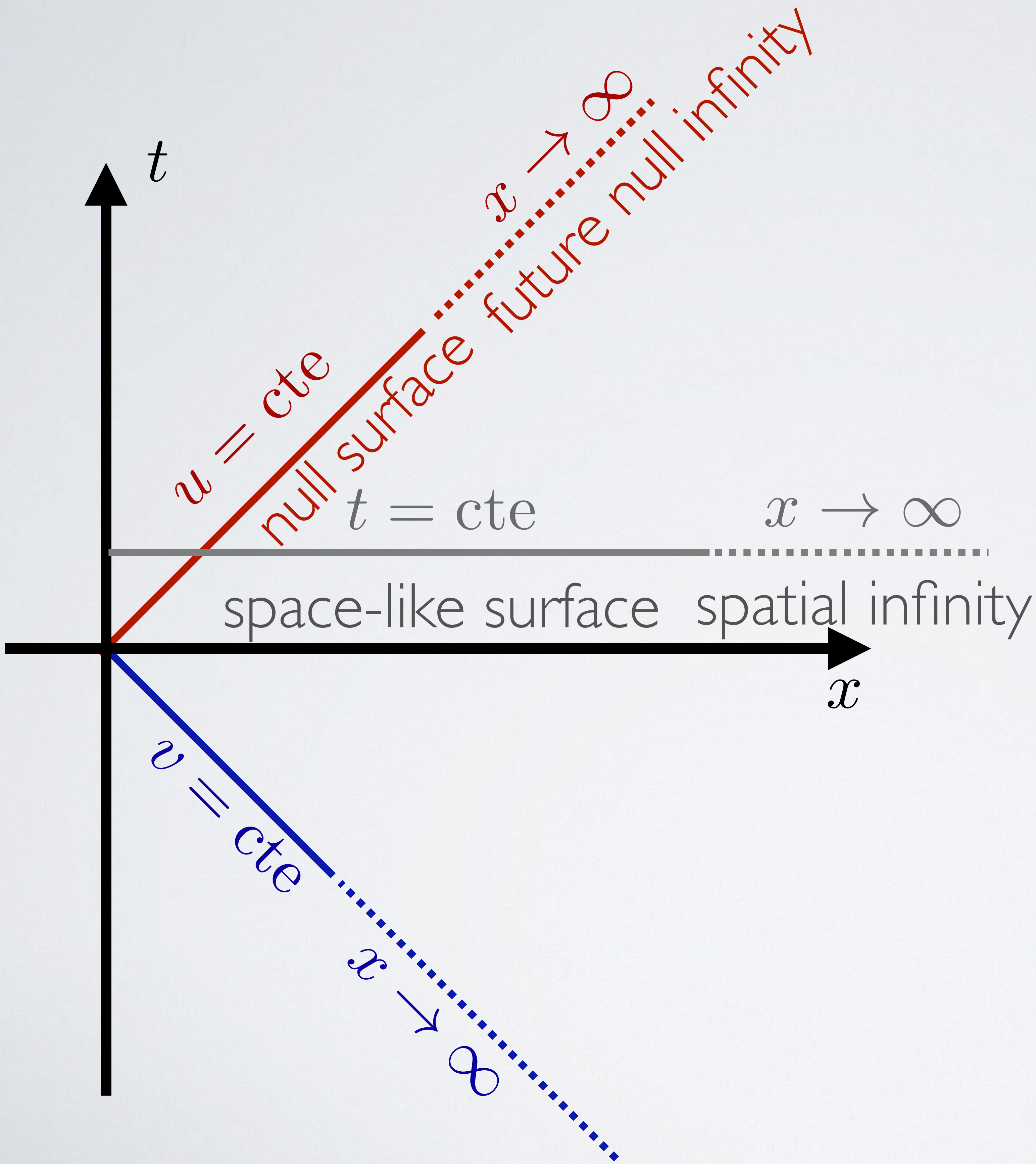

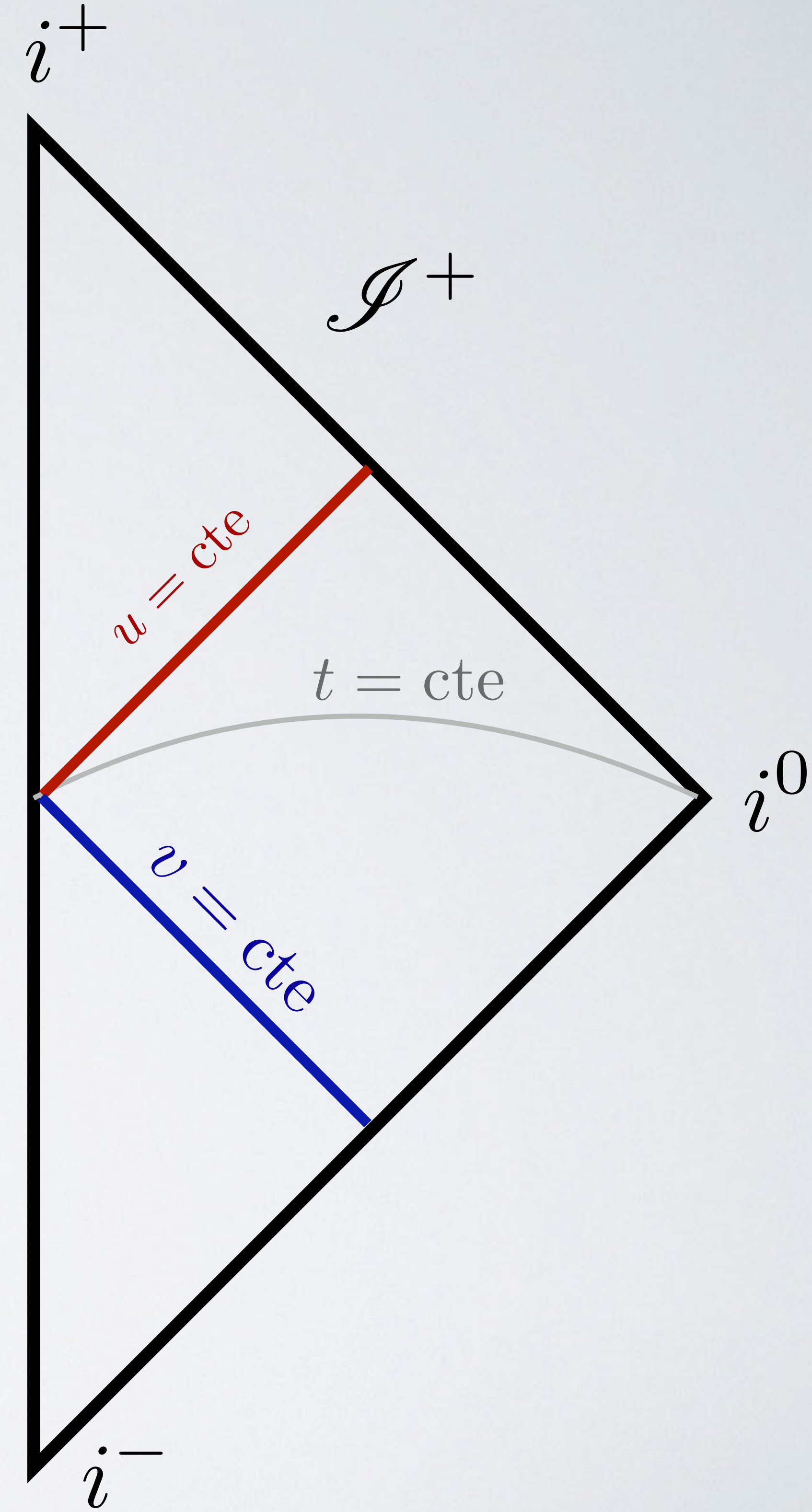

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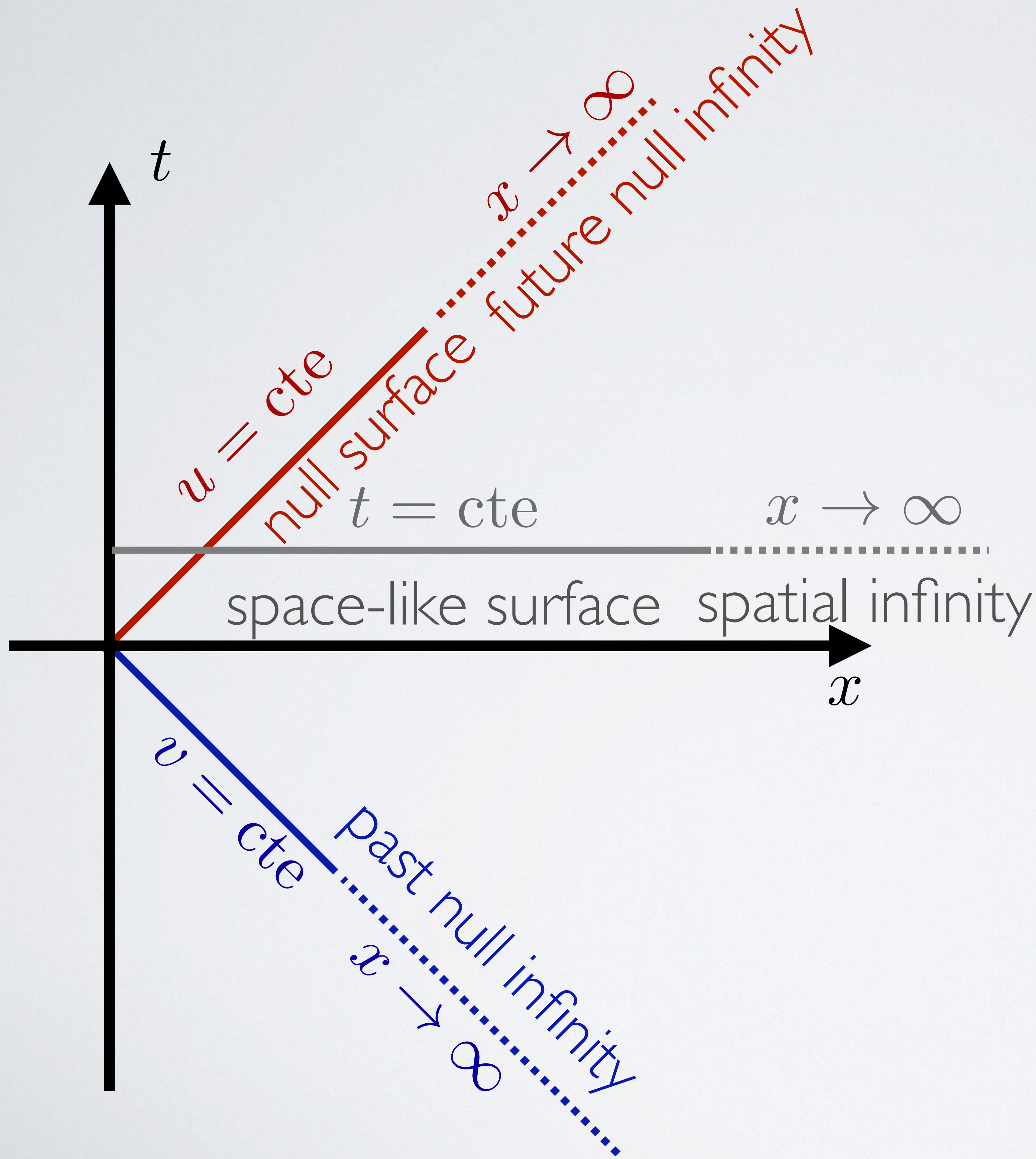

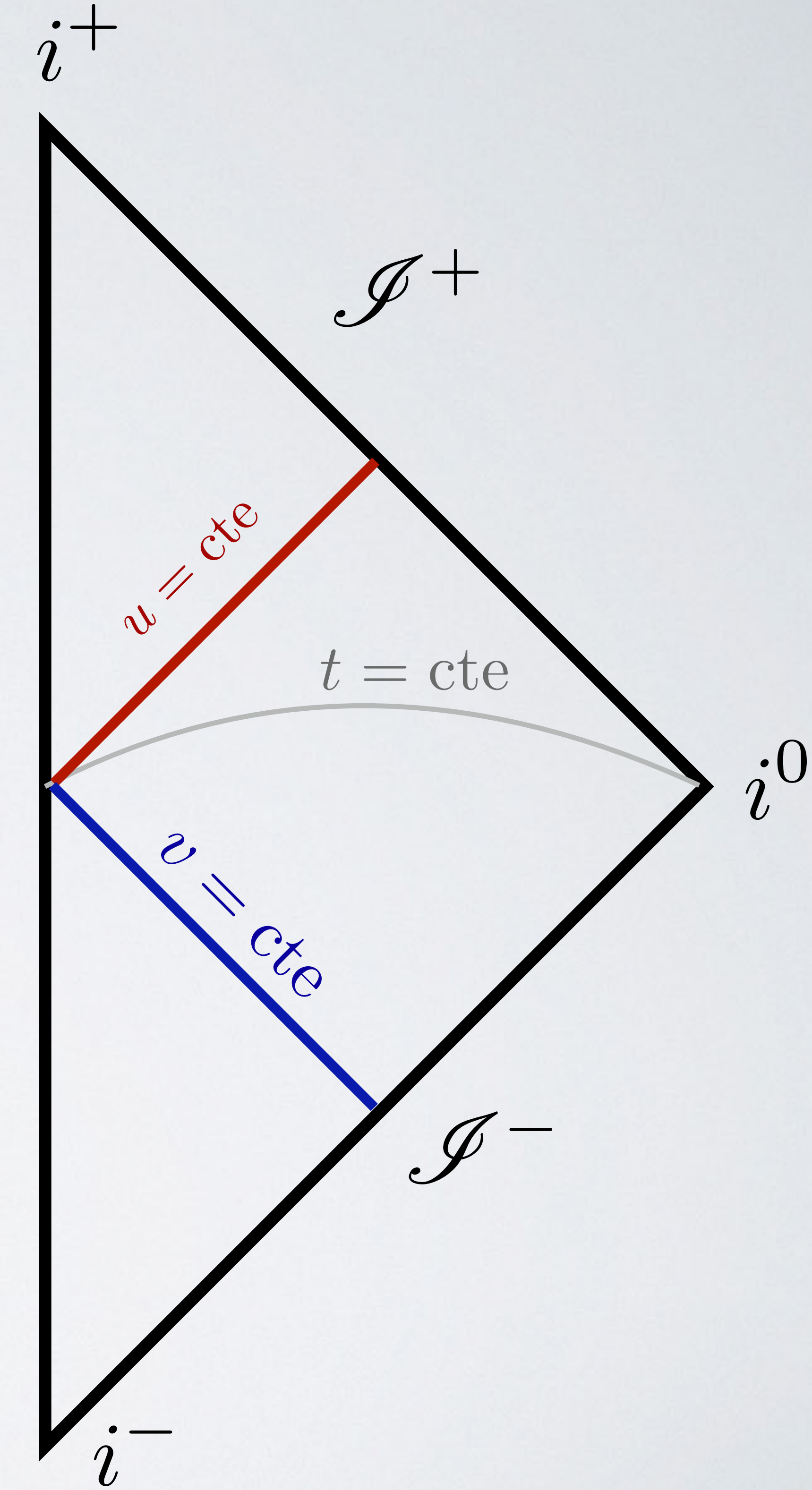

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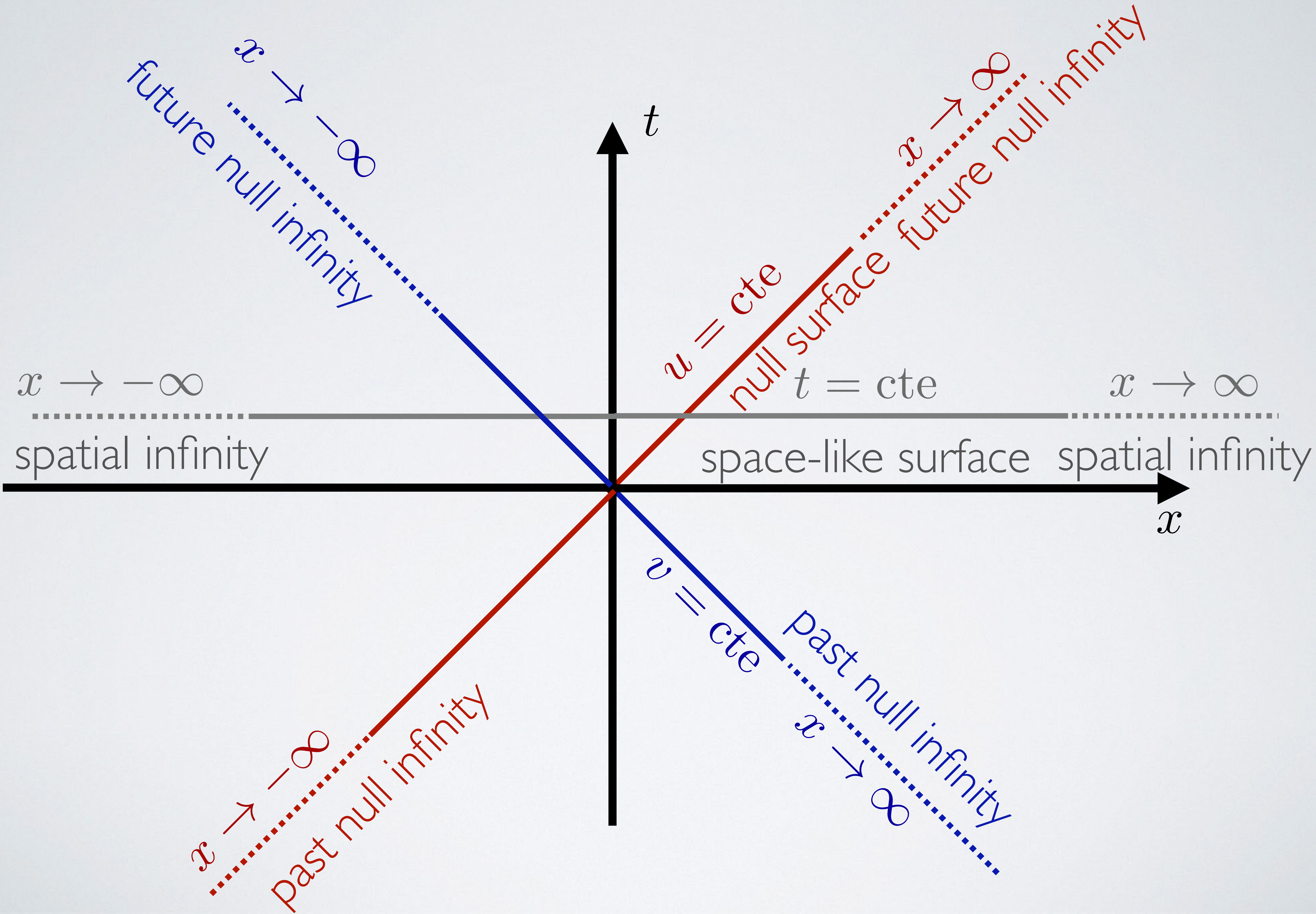

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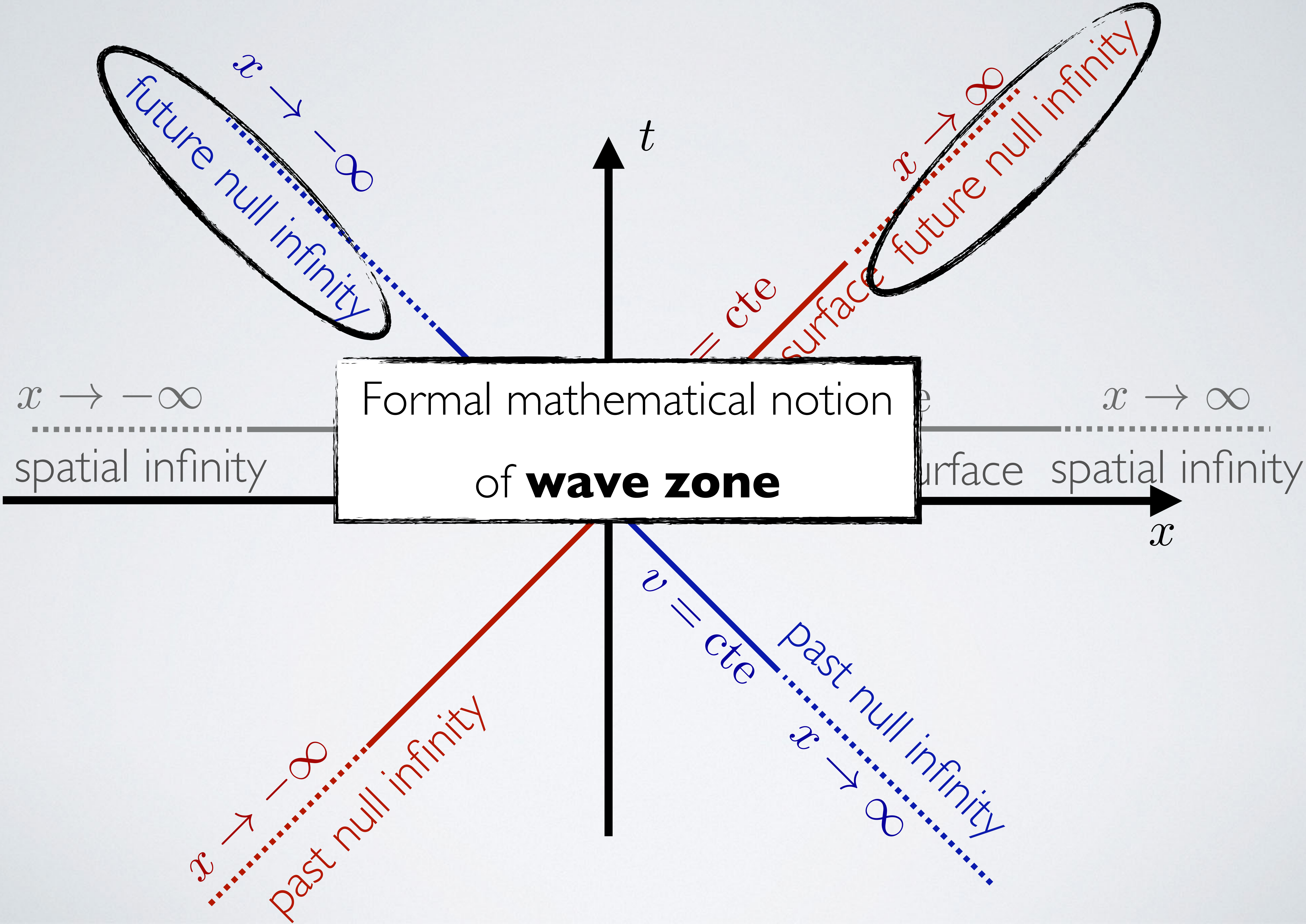
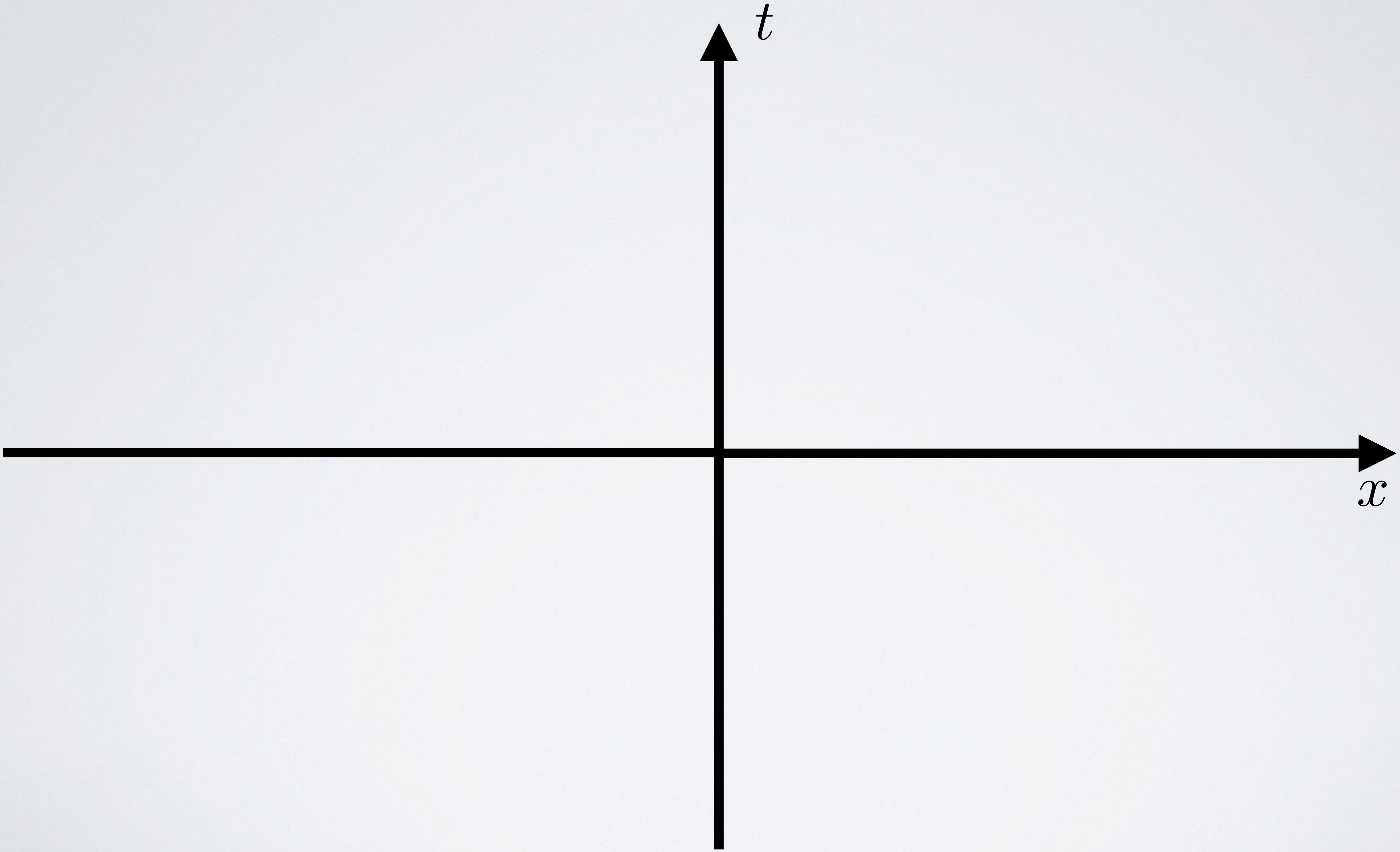


Formal mathematical notion

of **wave zone**

POTENTIAL



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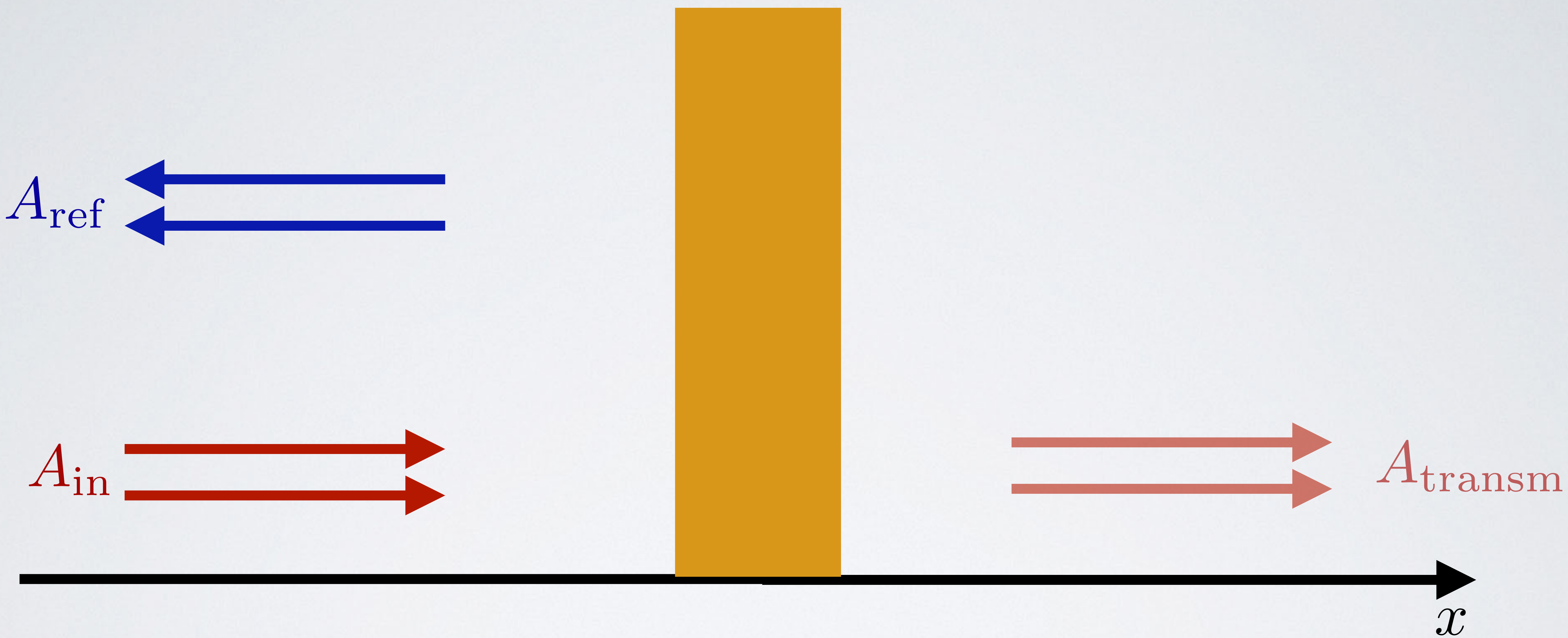

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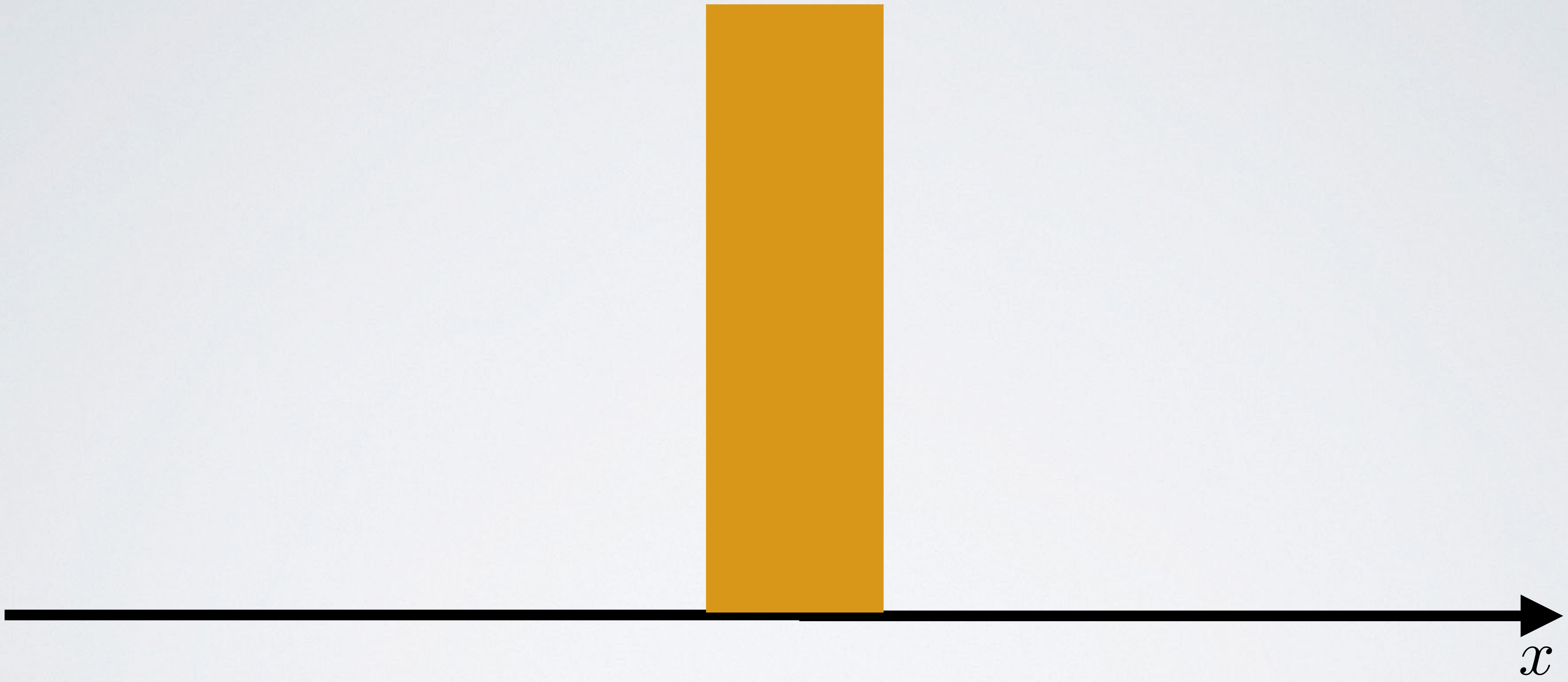

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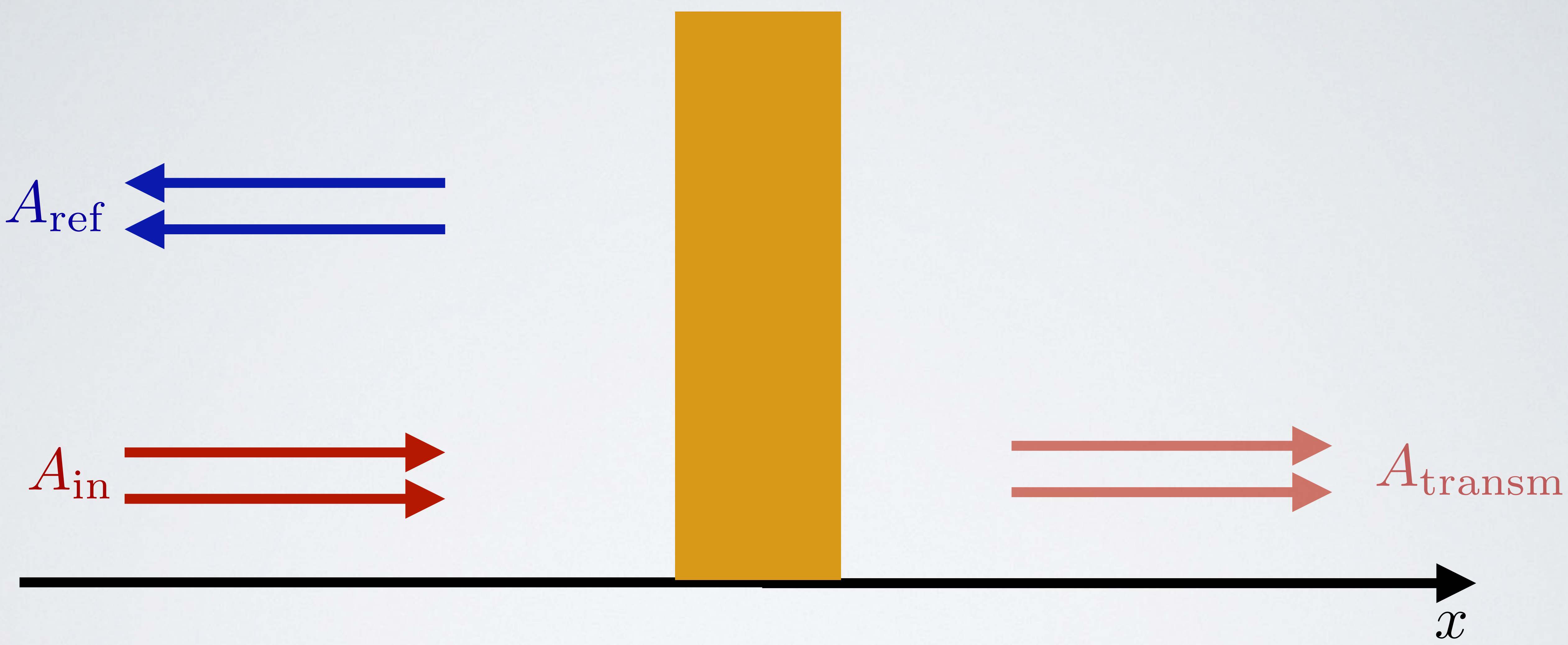

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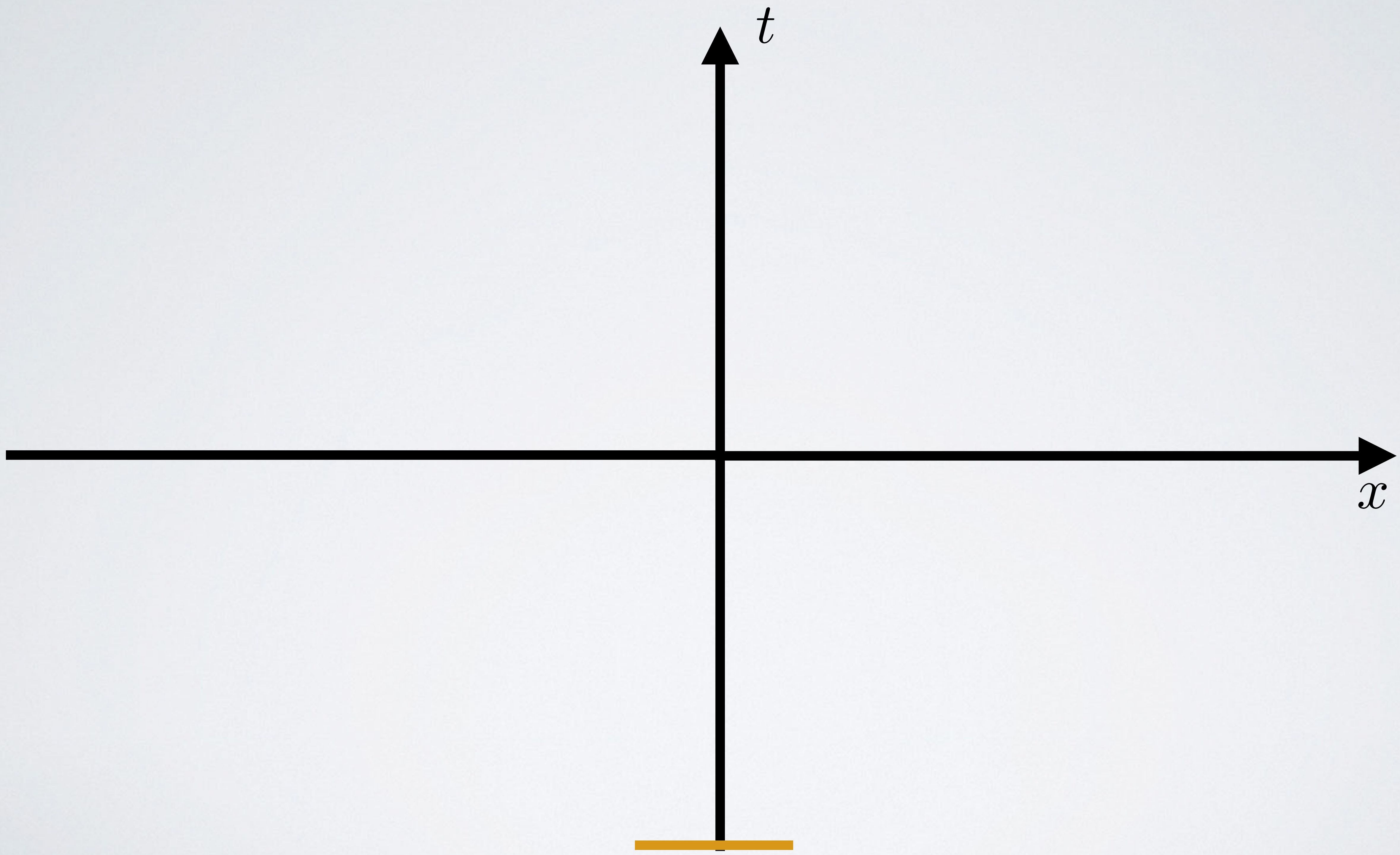

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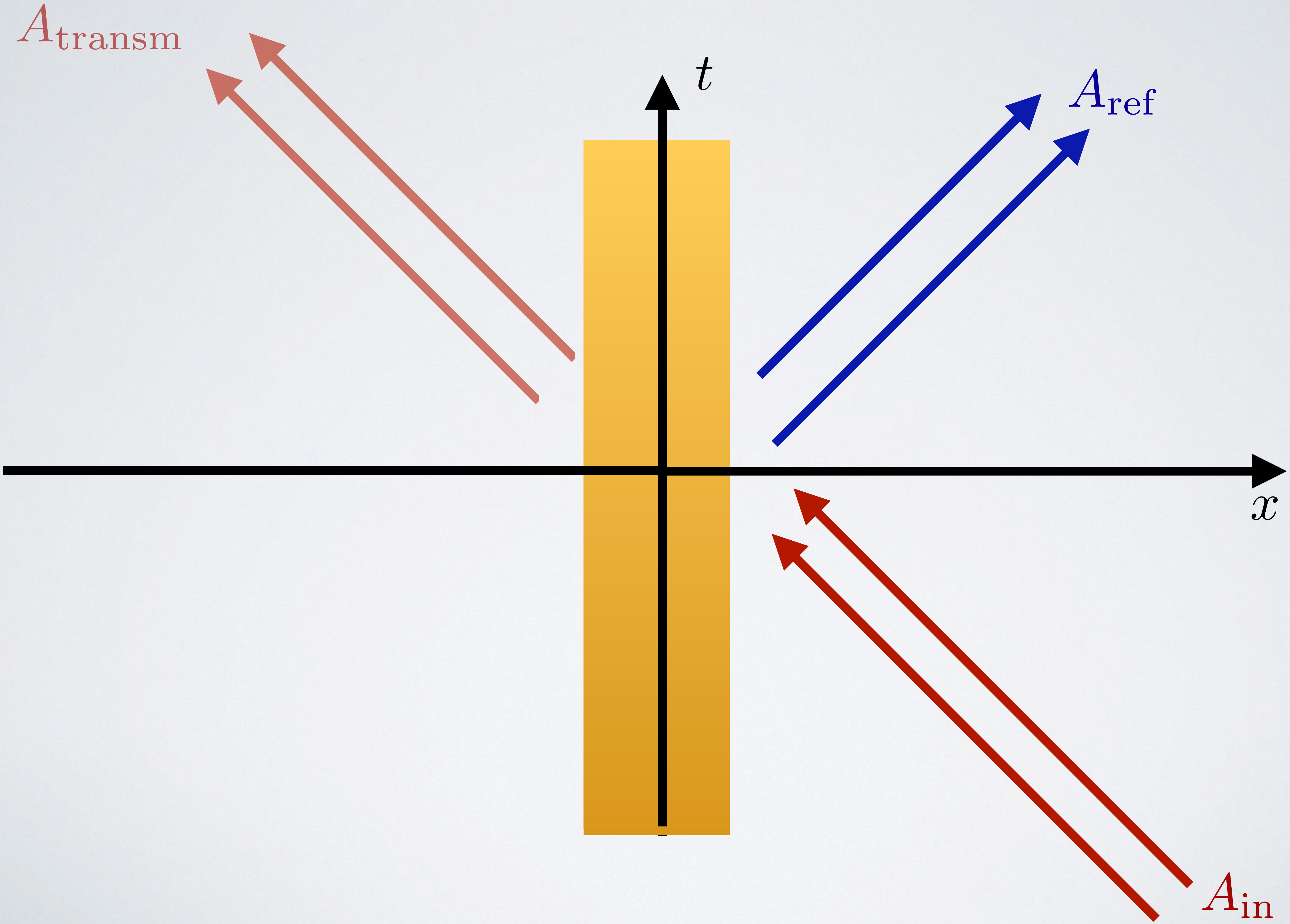

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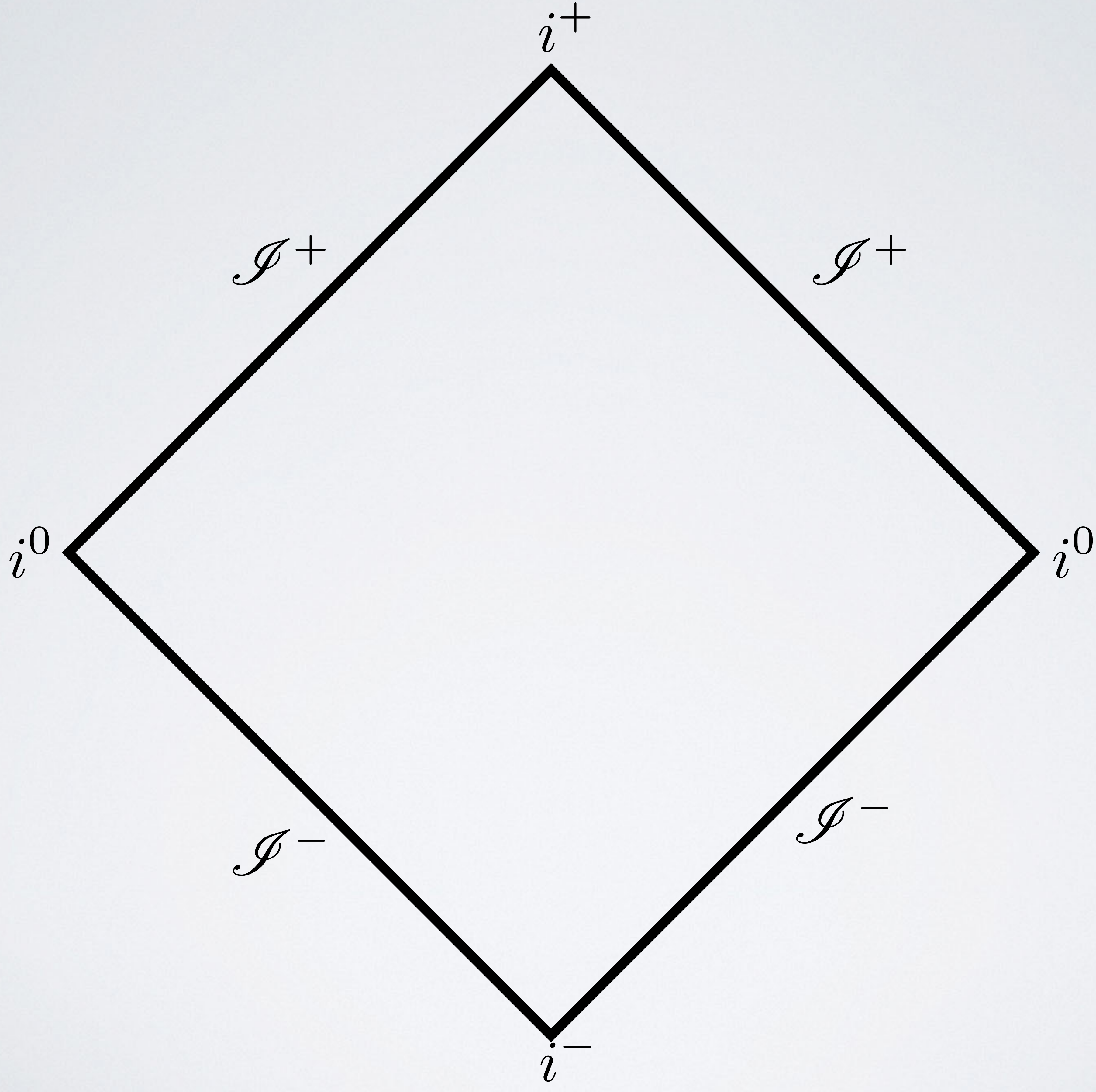

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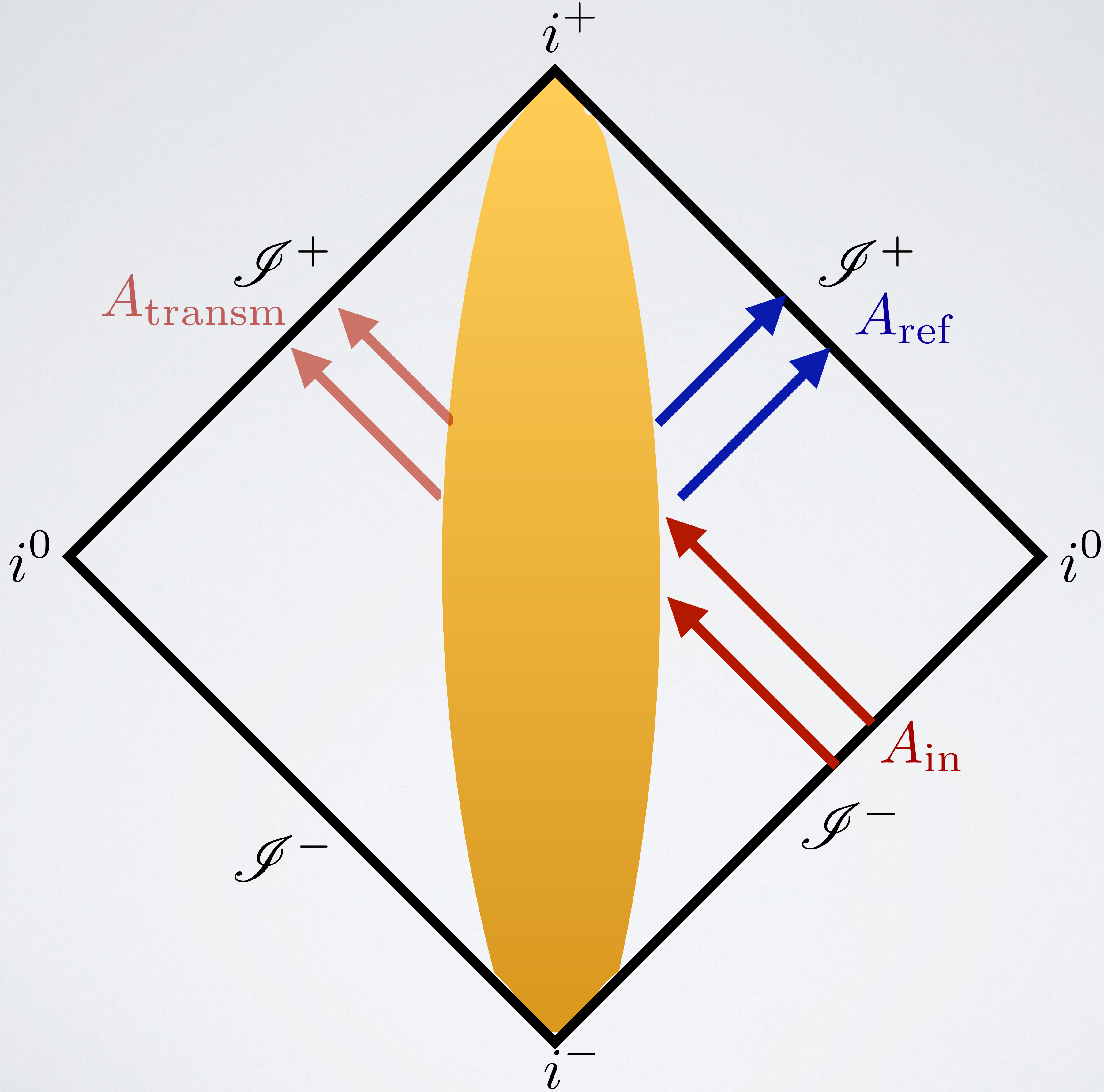

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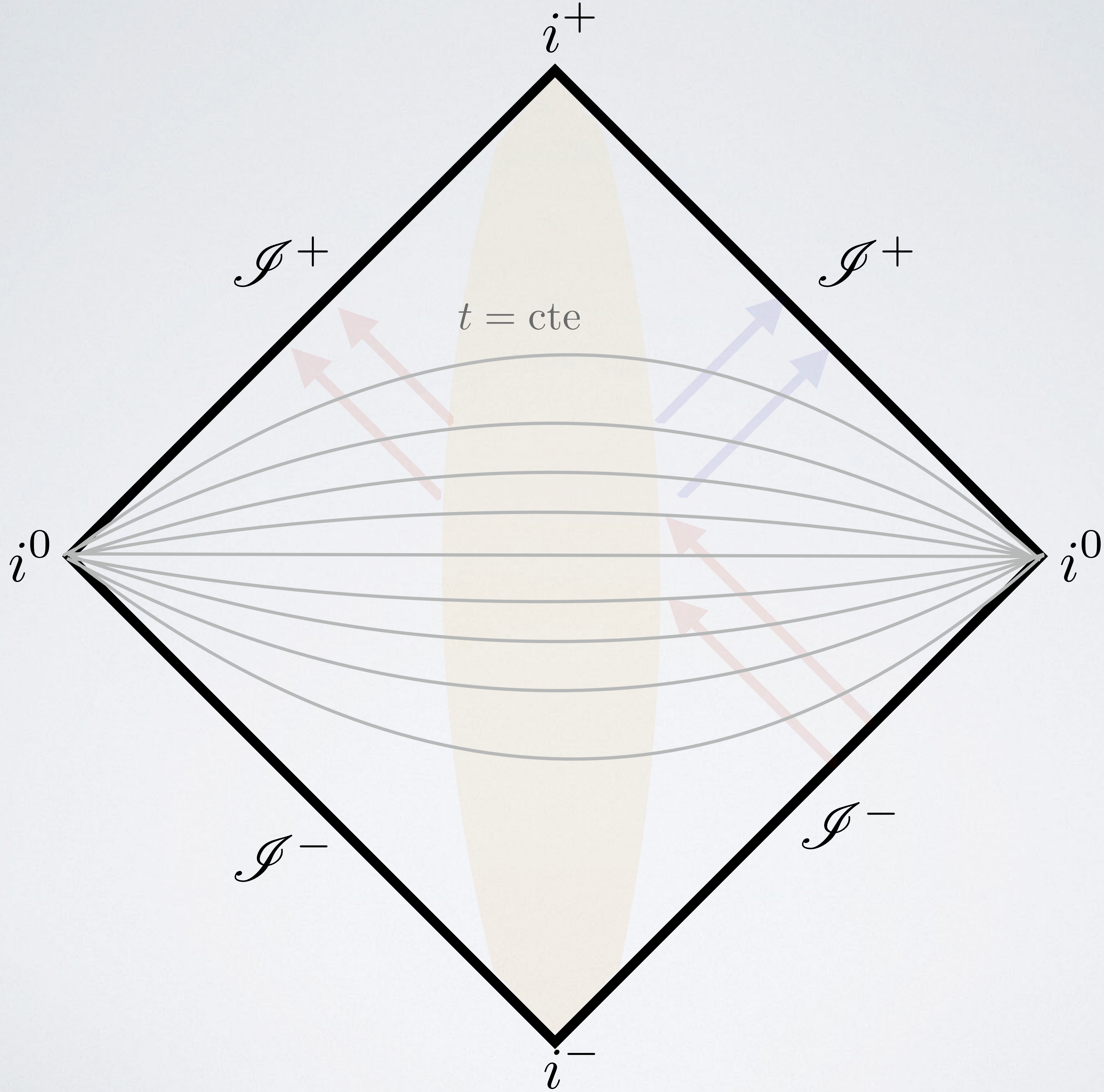

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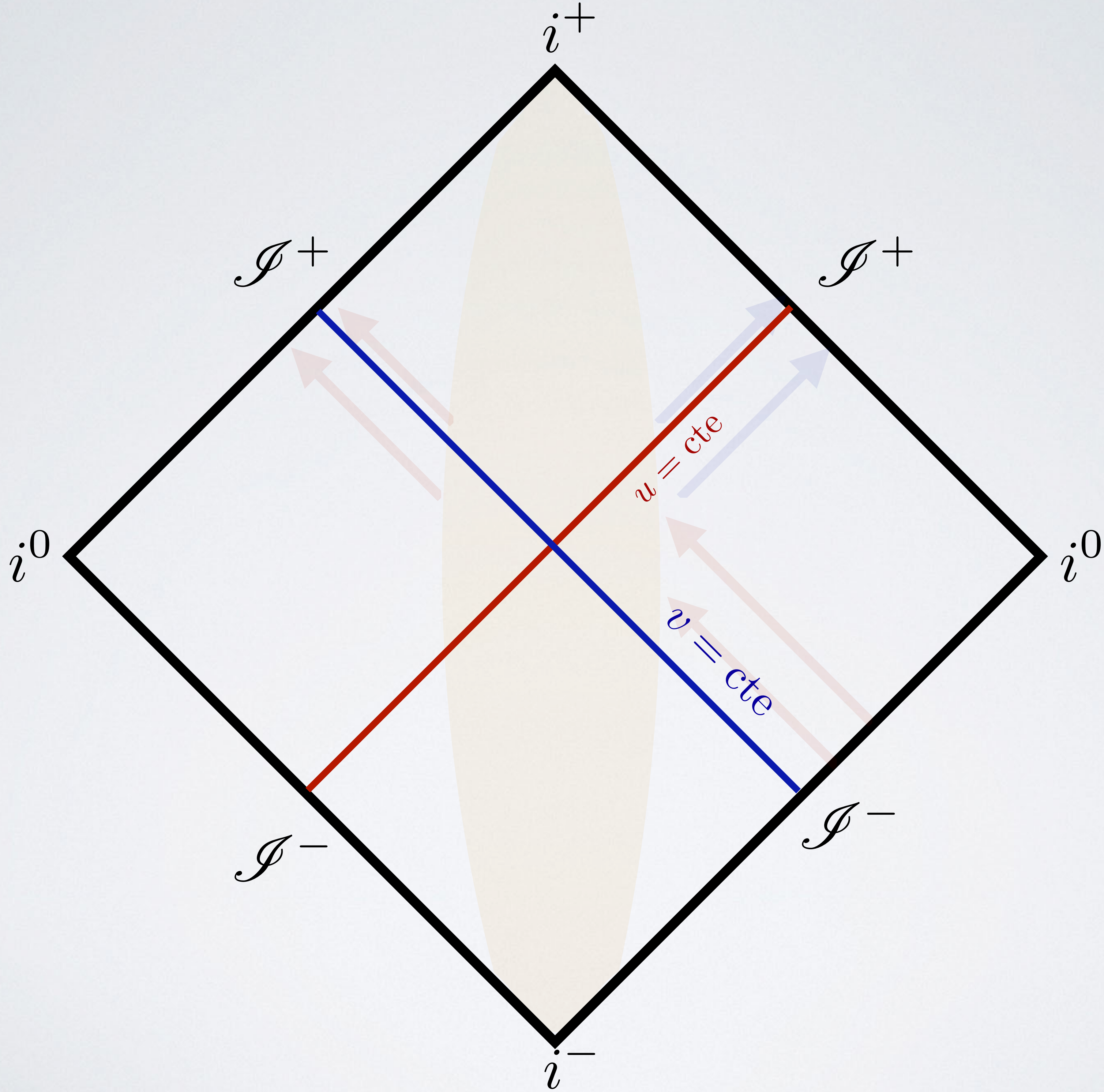

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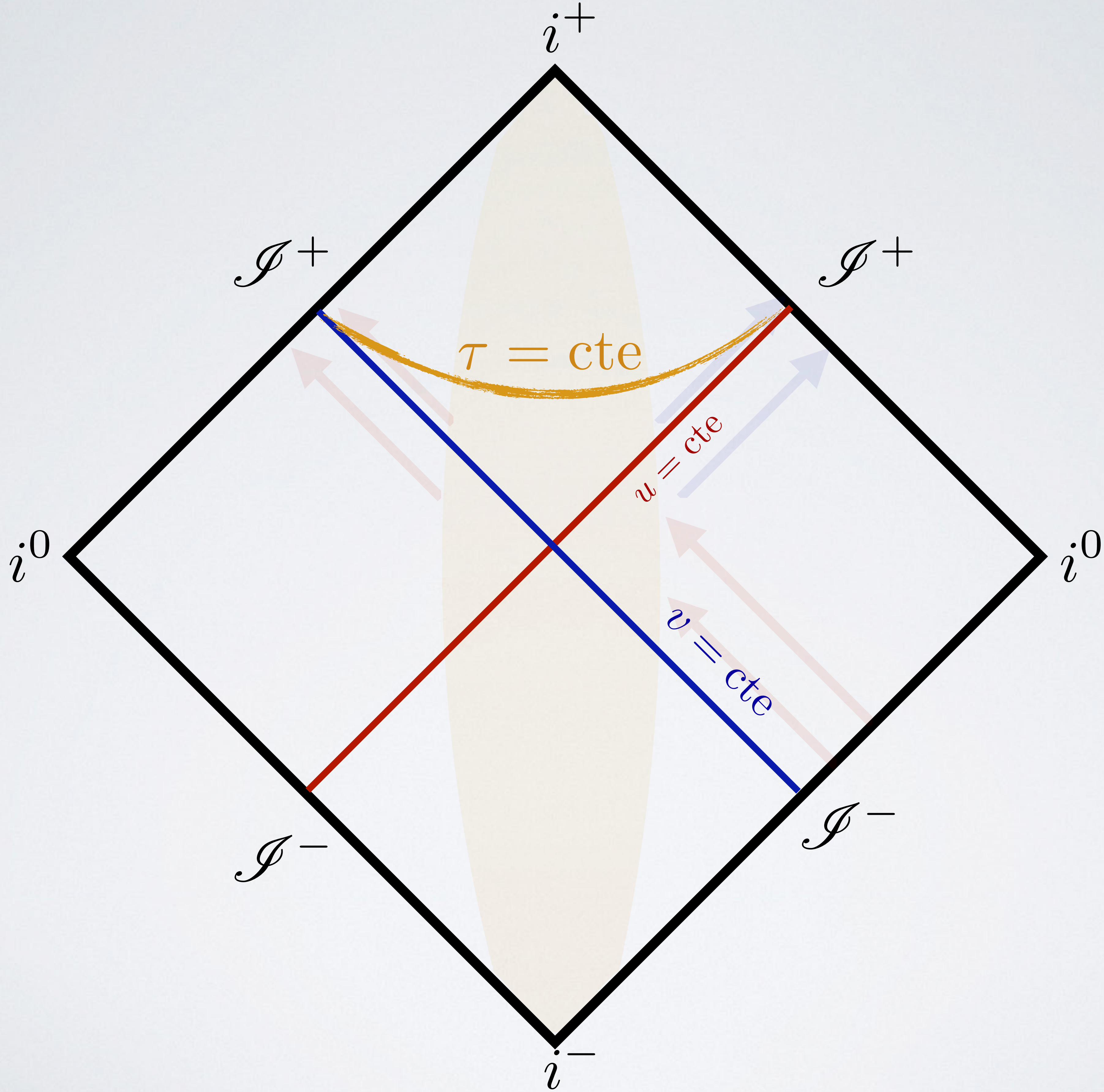

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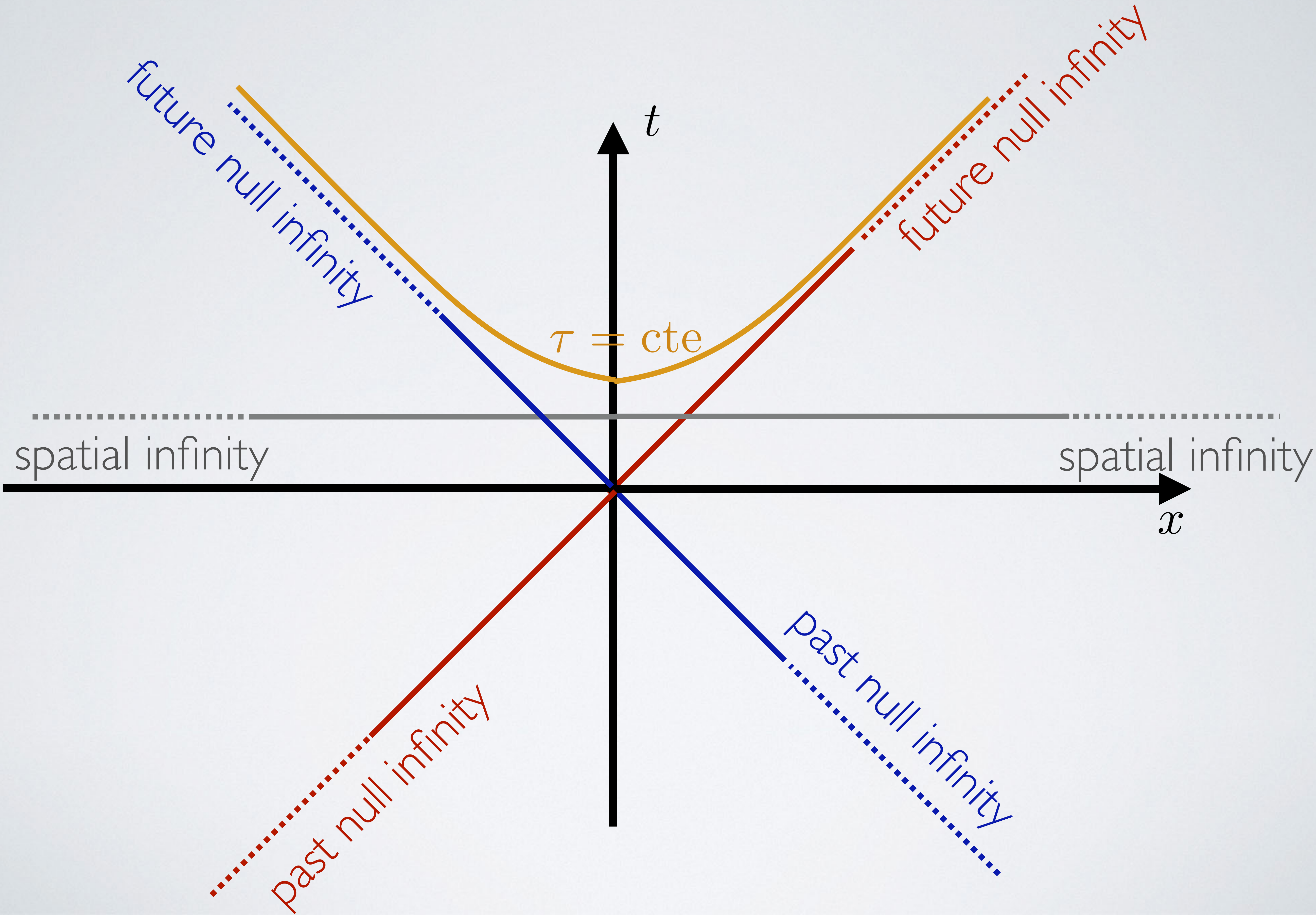
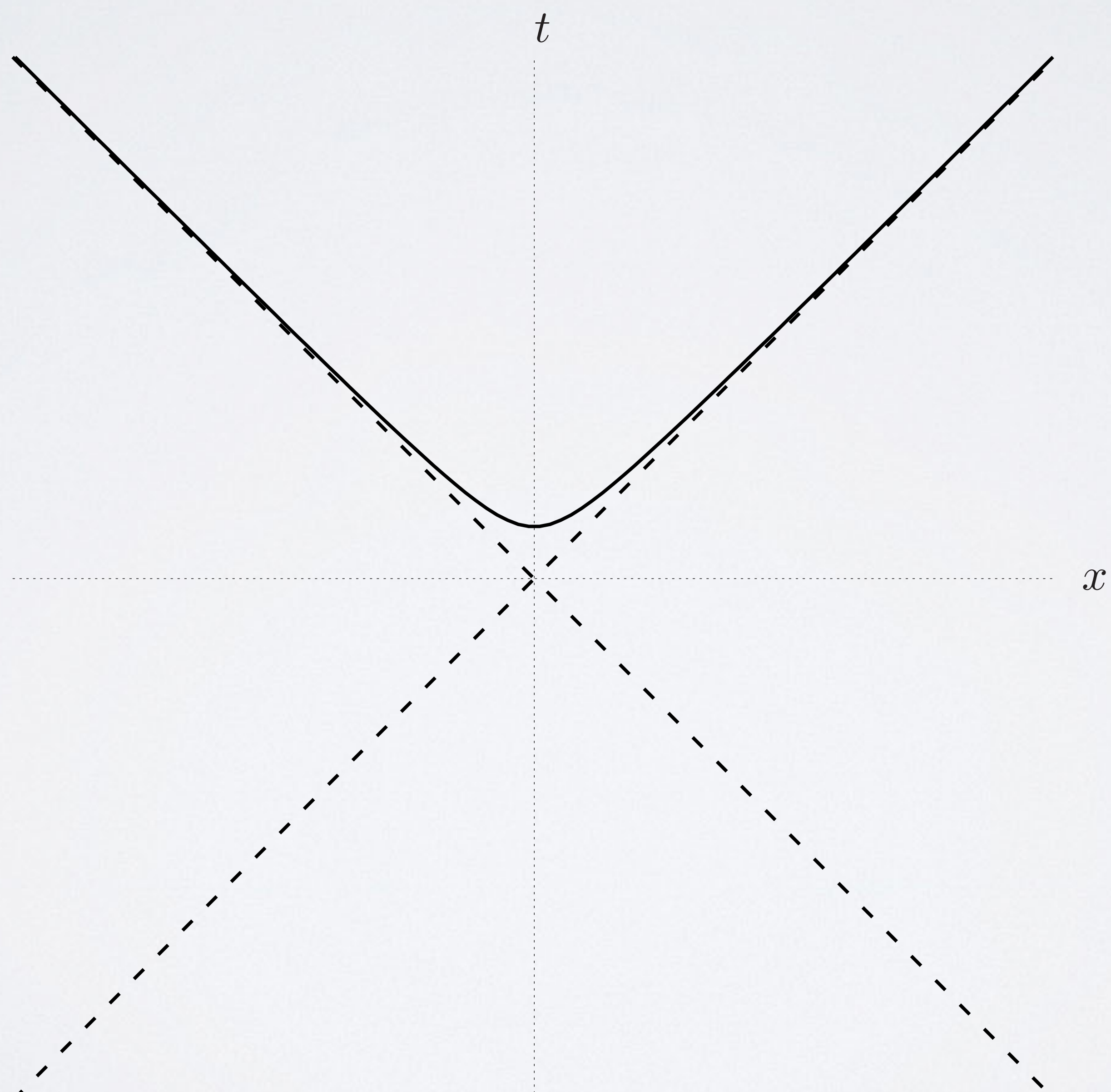


HYPERBOLOIDS

- Minkowski Spacetime: $ds^2 = -dt^2 + dx^2$
- Hyperboloid: $t^2 - x^2 = 1$



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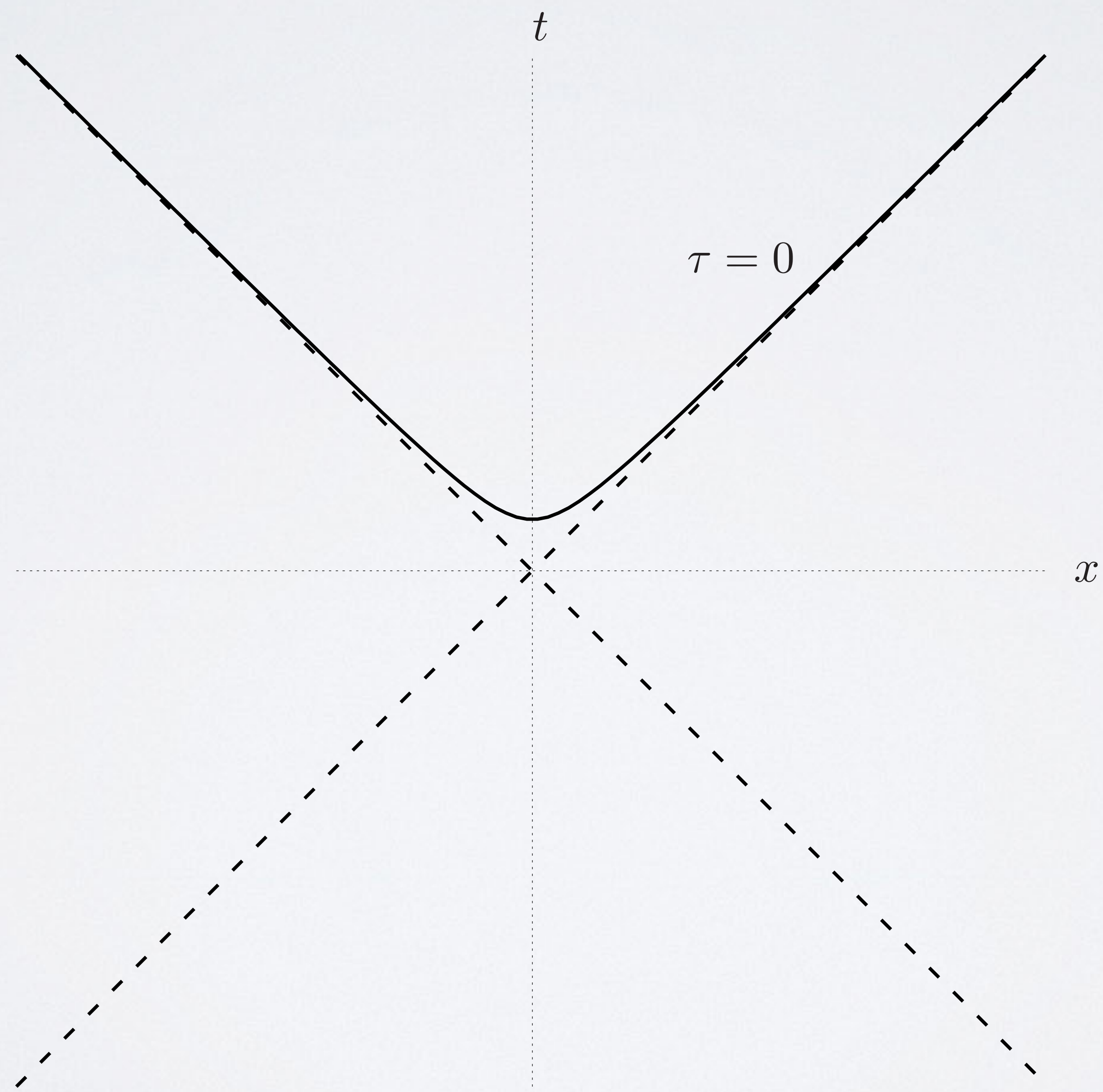

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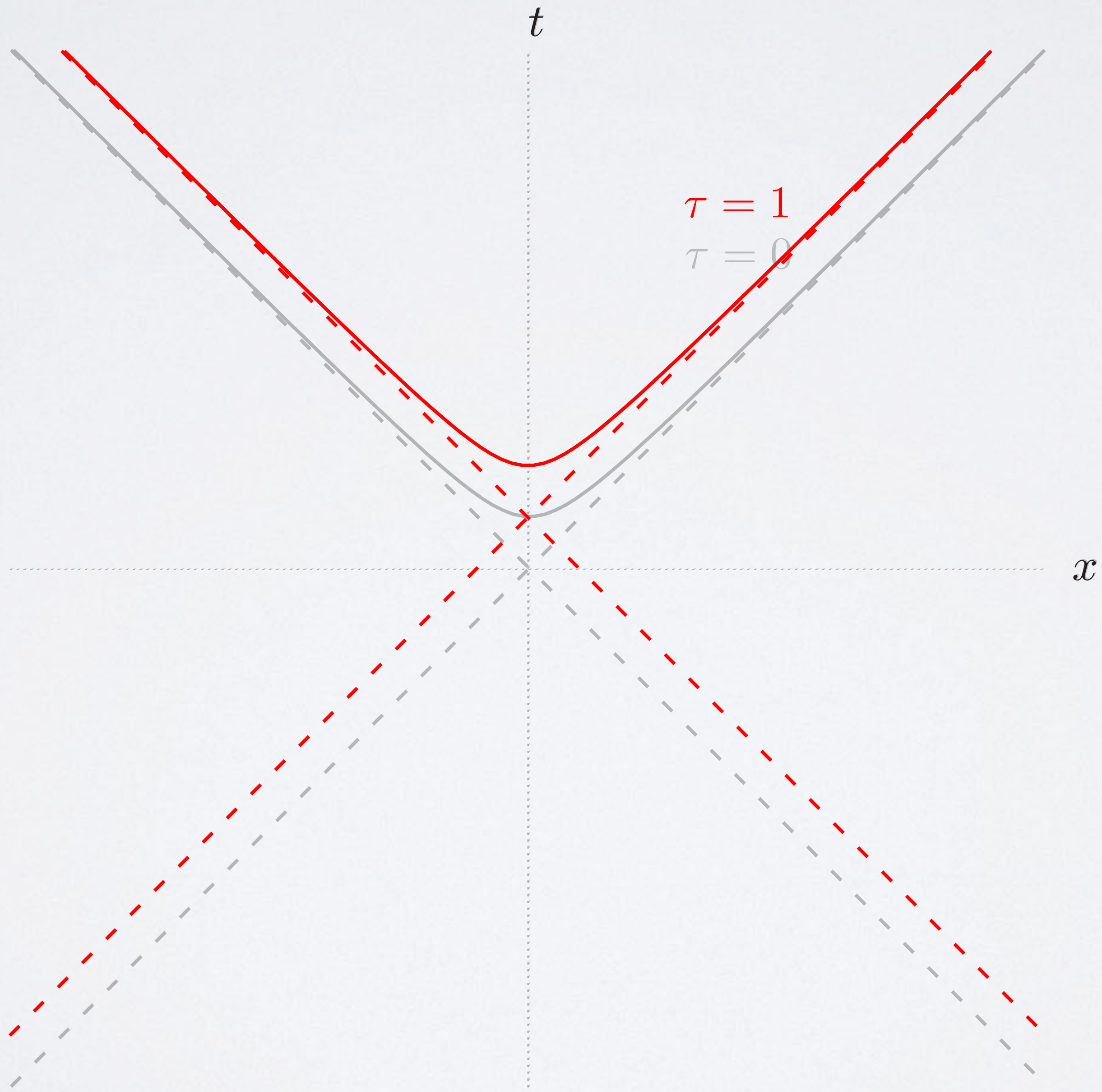

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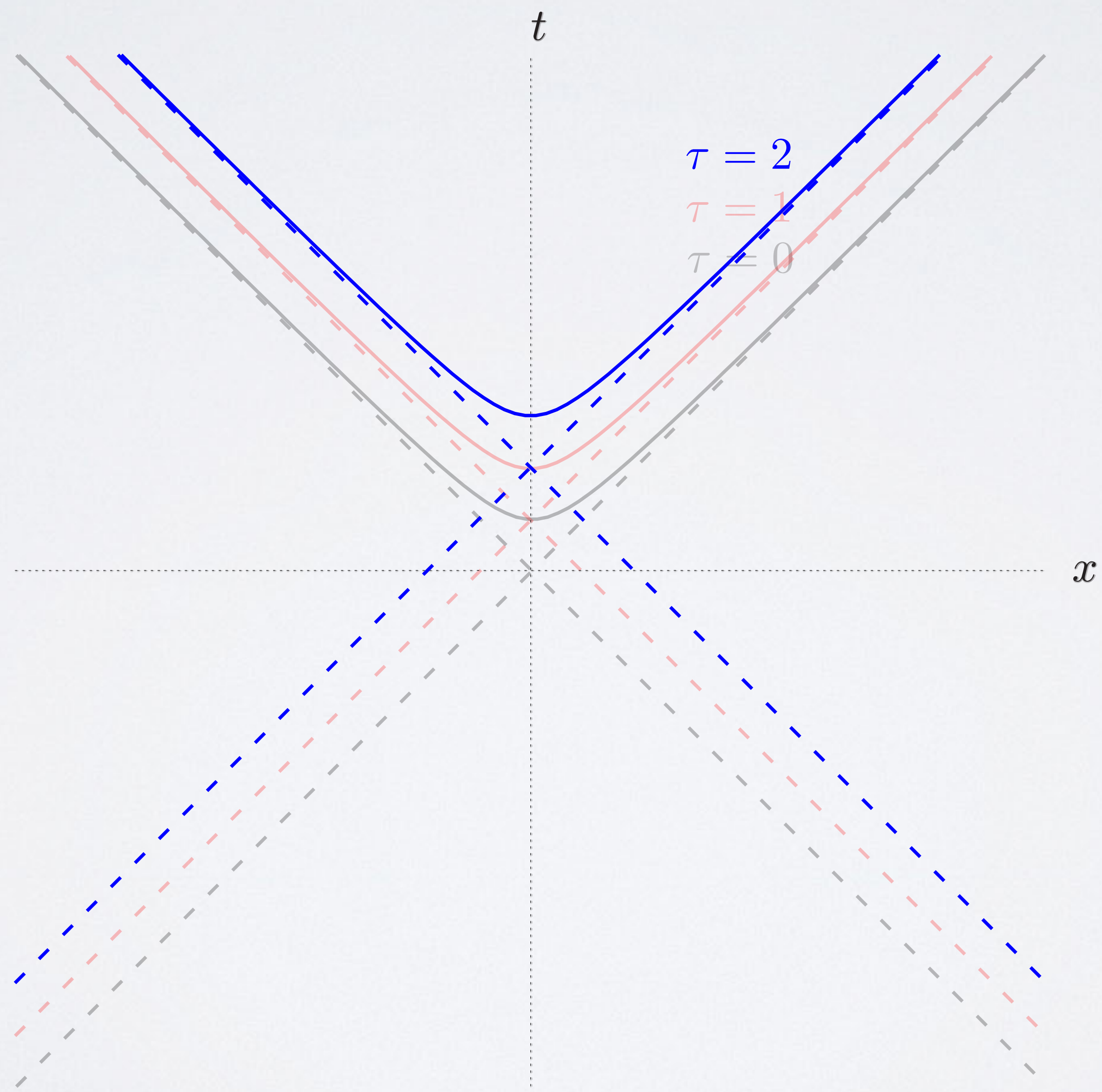

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Irregular at $\mathscr{I}^+$

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ds^2 → Physical metric, Irregular at $\mathscr{I}^+$

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$d\tilde{s}^2$: "non-physical" metric regular at $\mathscr{I}^+$

$$\Omega = \cos(y)$$

$$d\tilde{s}^2 = -\cos^2(y) d\tau^2 - 2\sin(y) d\tau dy + dy^2$$

HYPERBOLOIDS

- Fundamental Structure in Lorentzian Geometry

[Anil Zenginoglu]

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Riemannian Geometry

HYPERBOLOIDS

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Riemannian Geometry ⟷ Spherical Coordinates

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HYPERBOLOIDS

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Riemannian Geometry ⟷ Spherical Coordinates

$$ds^2 = dx^2 + dy^2$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

HYPERBOLOIDS

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[Anil Zenginoglu]

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Lorentzian Geometry

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Lorentzian Geometry ⟷ Hyperboloidal Coordinates

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HYPERBOLOIDS

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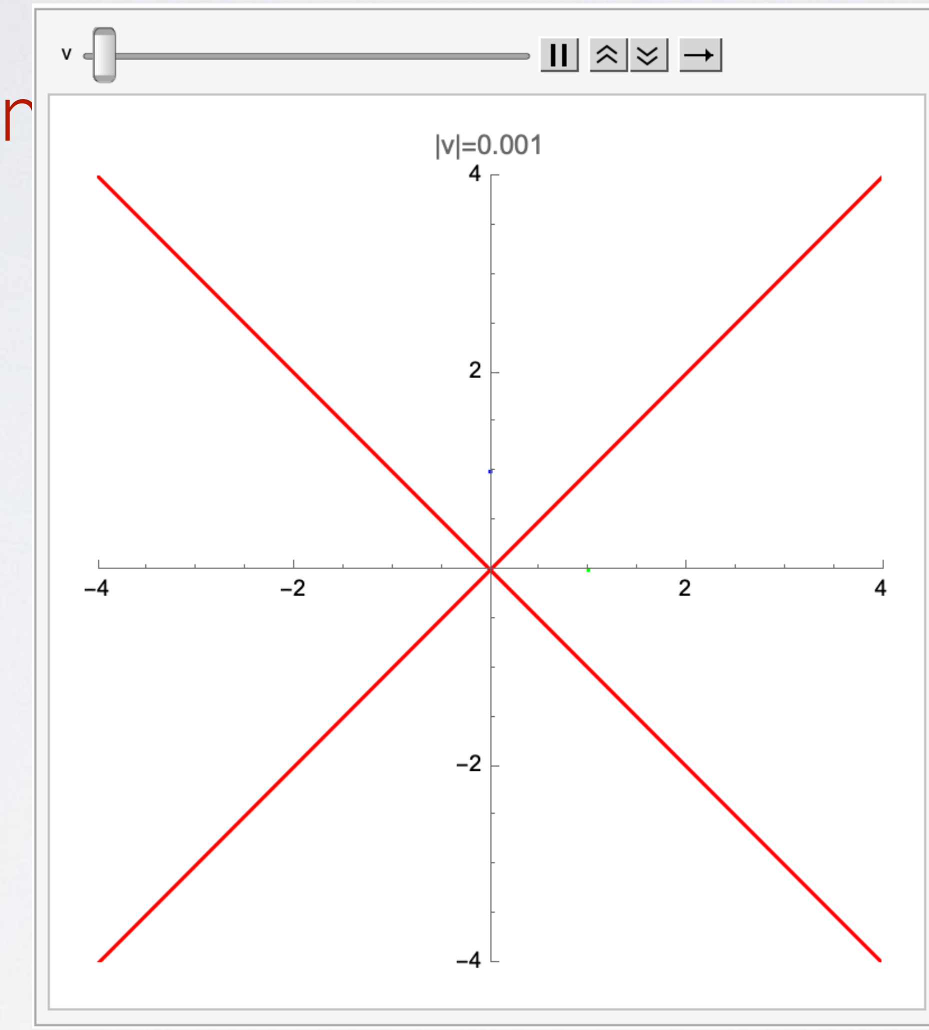

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$$v = \tanh\gamma$$

BLACK HOLE SPACETIME

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \qquad f(r) = 1 - \frac{r_h}{r}$$

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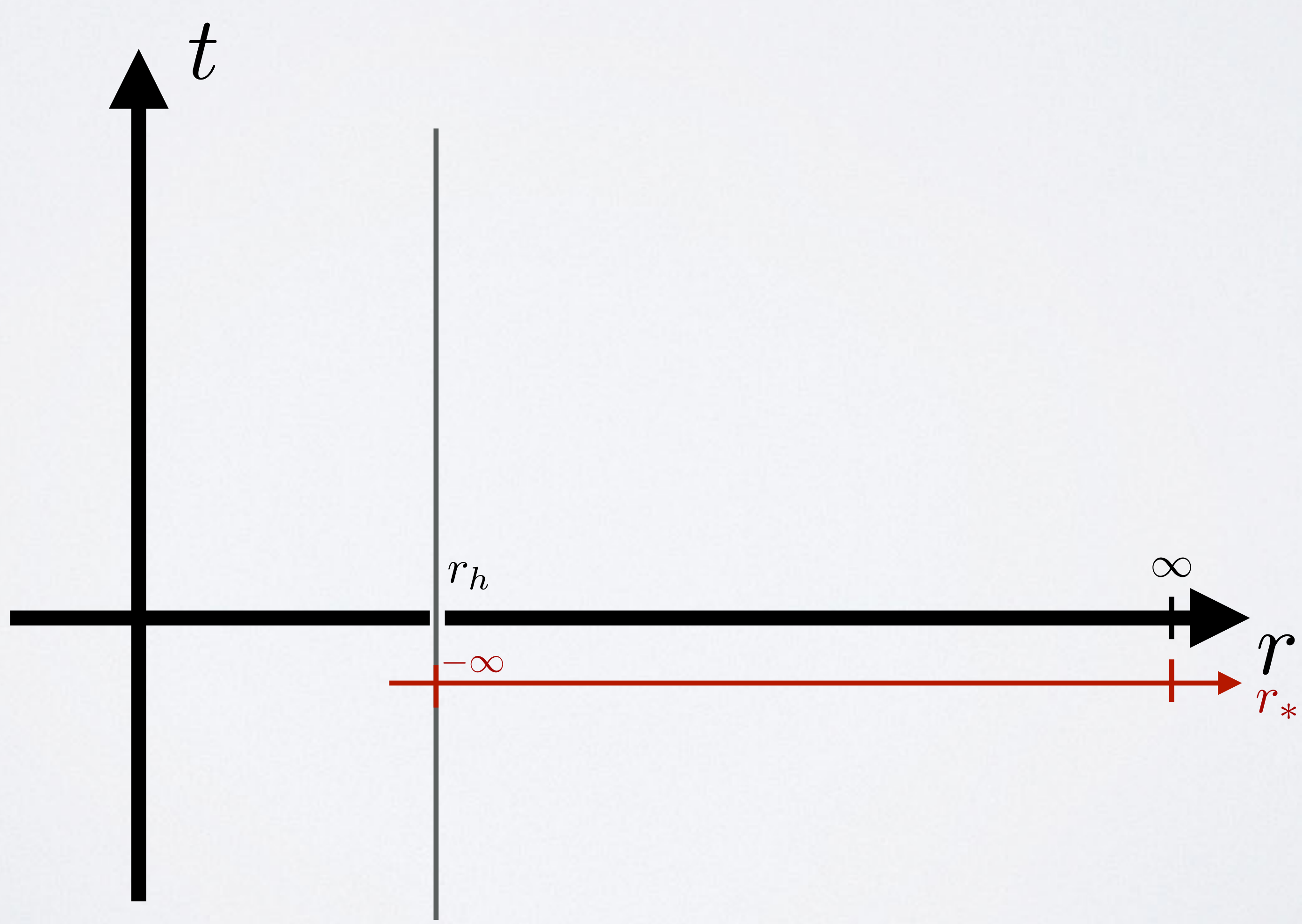
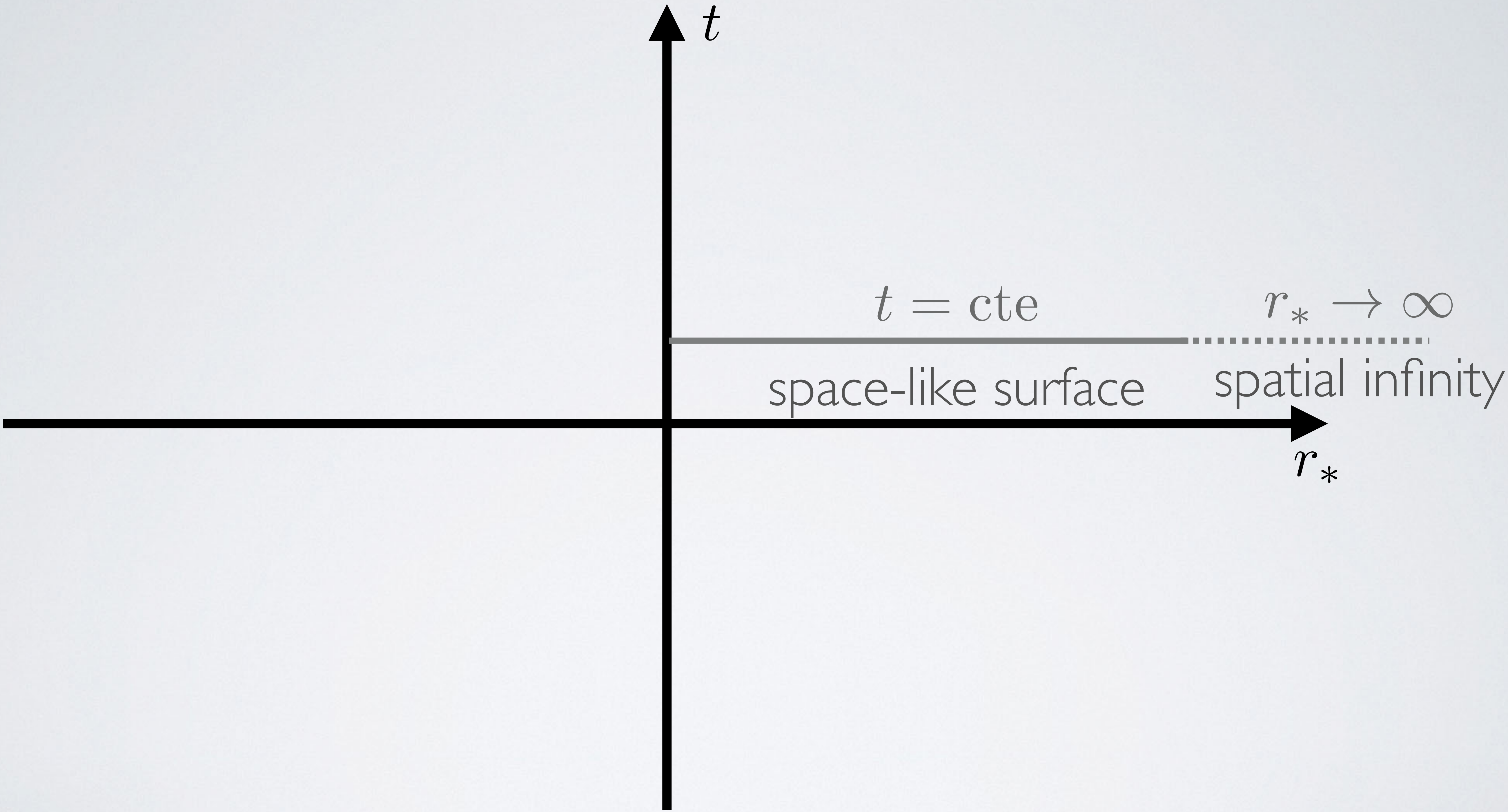
BLACK HOLE SPACETIME

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \qquad f(r) = 1 - \frac{r_h}{r}$$

$$ds^2 = f(r)\left(-dt^2 + dr_*^2\right) + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \qquad \frac{dr_*}{dr} = \frac{1}{f(r)}$$



COORDINATES



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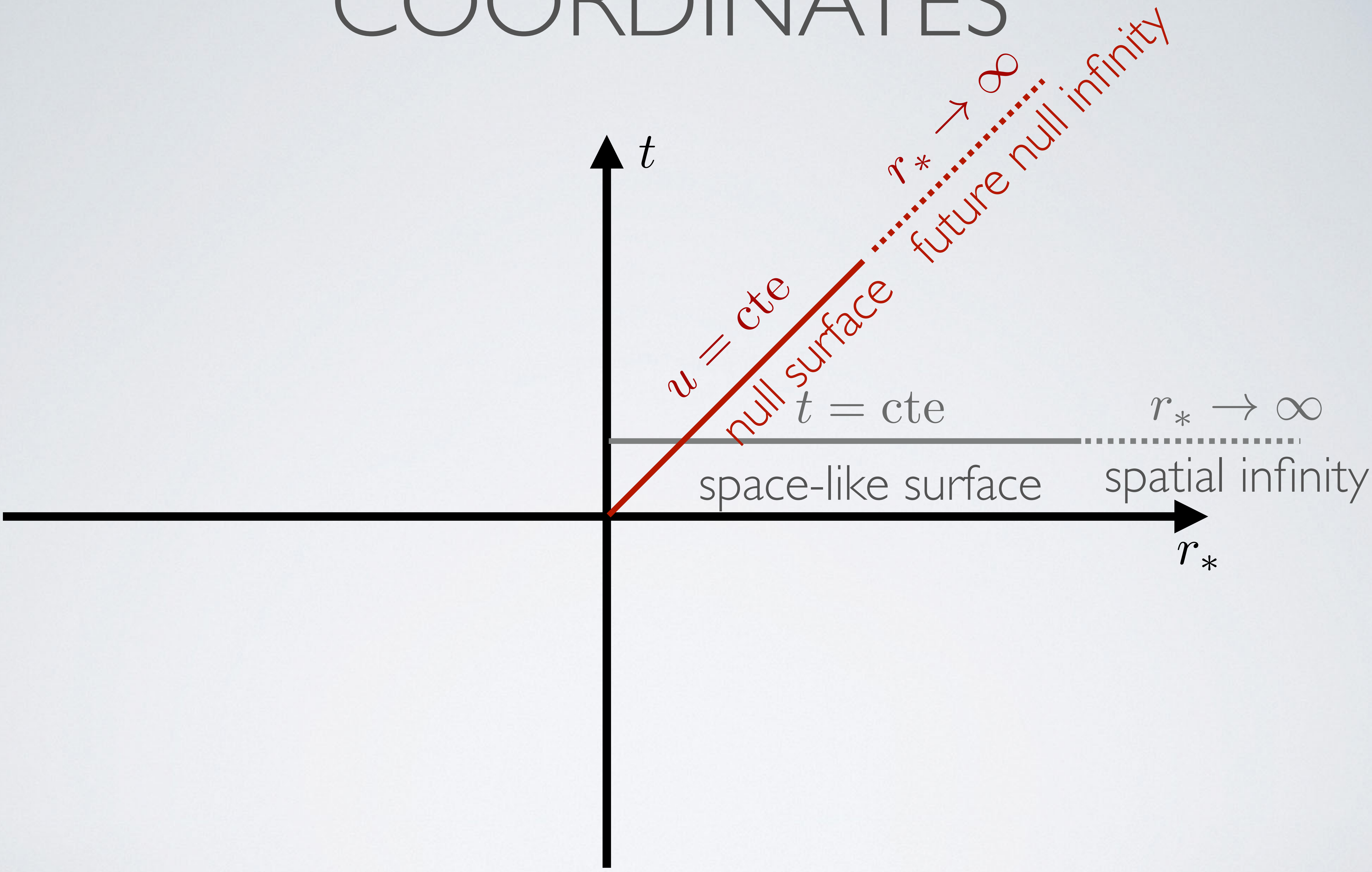

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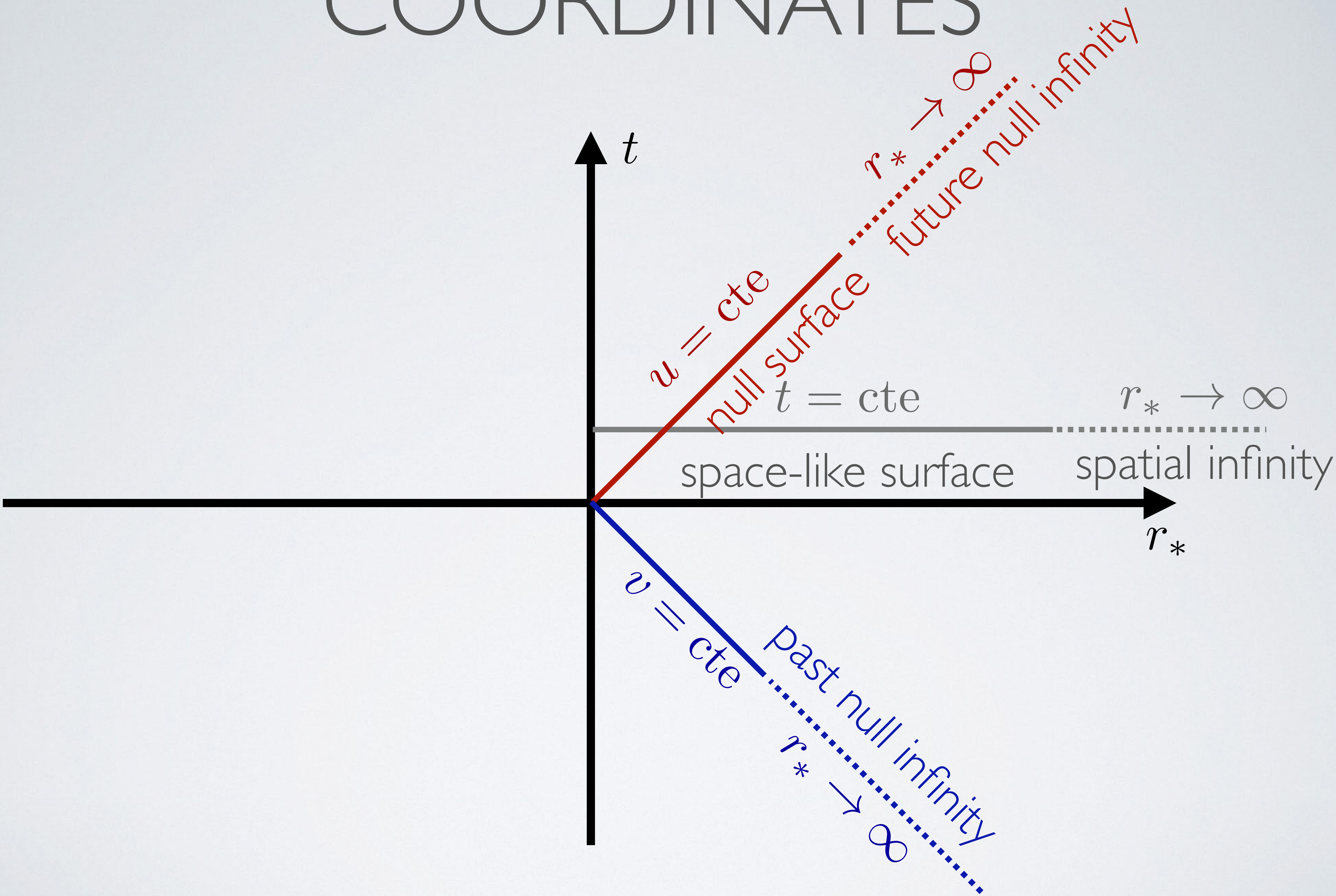

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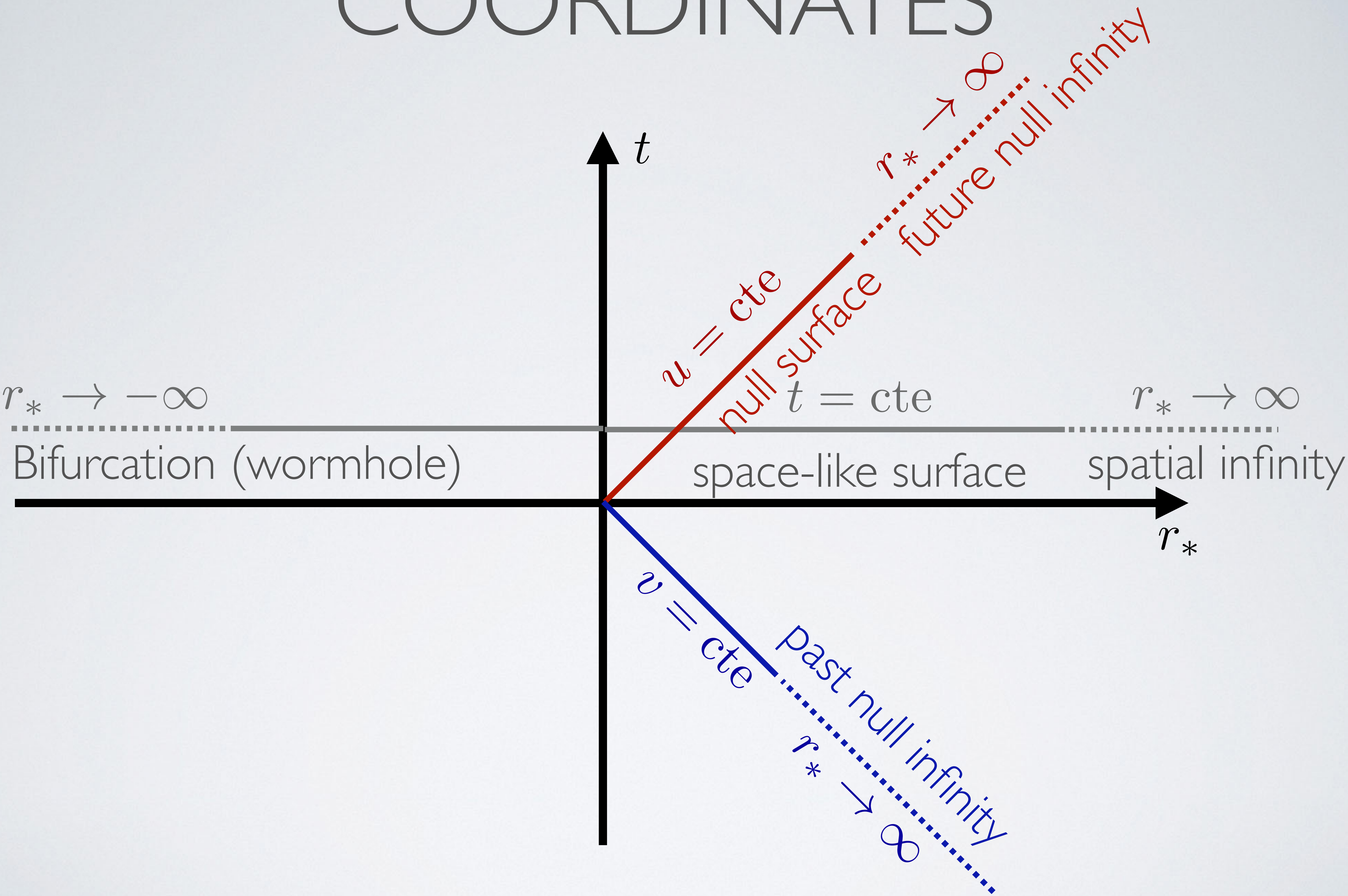

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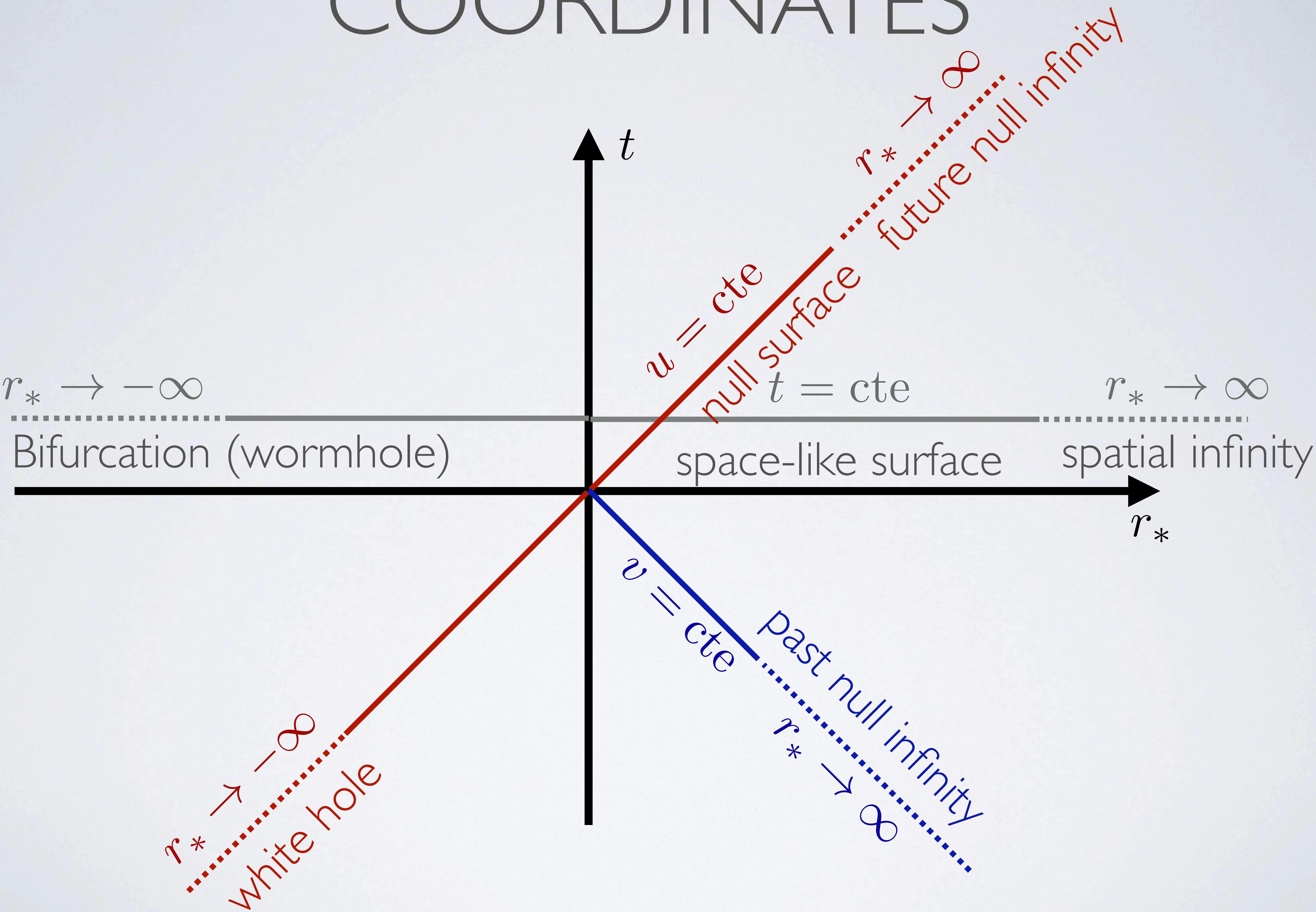

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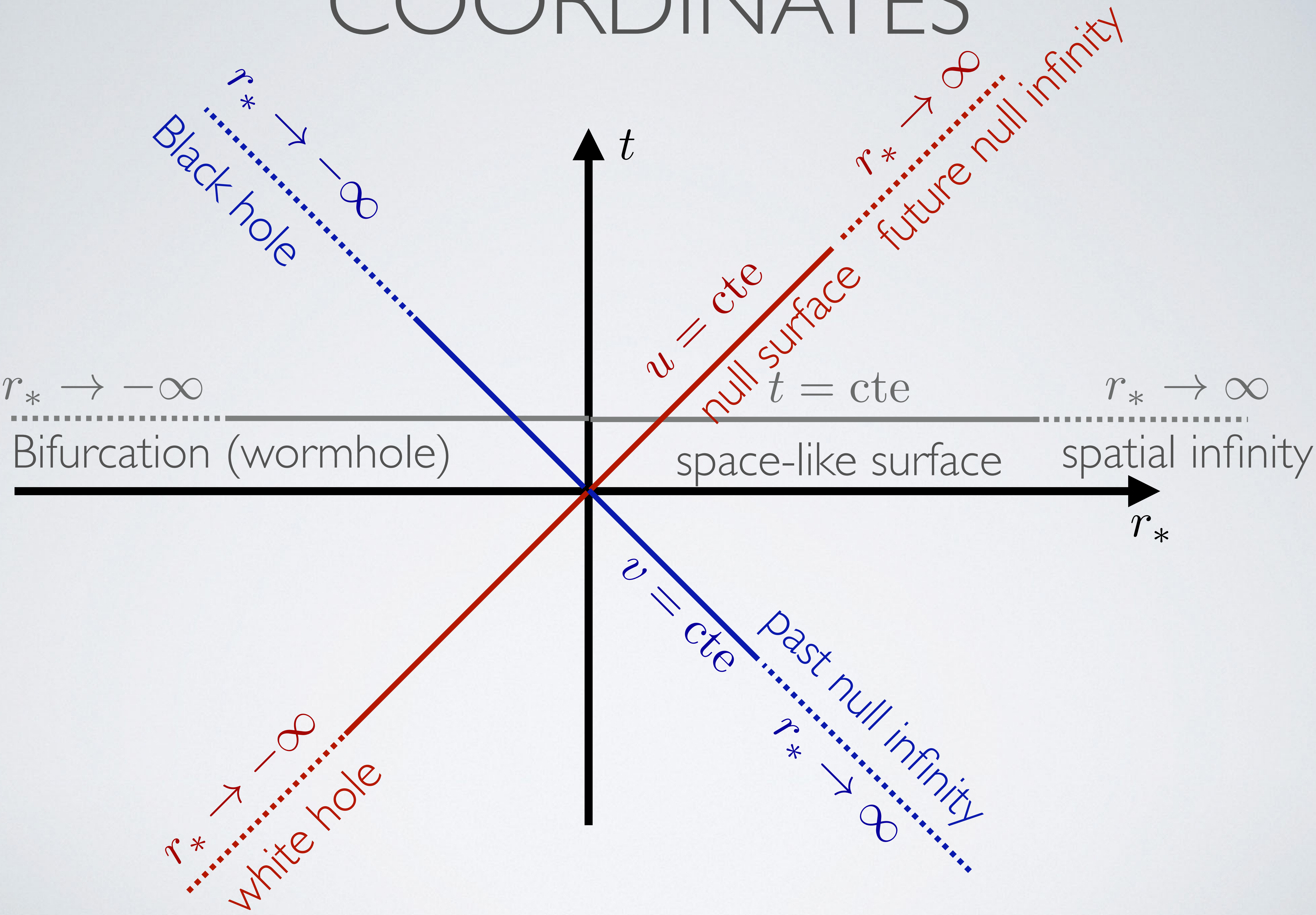

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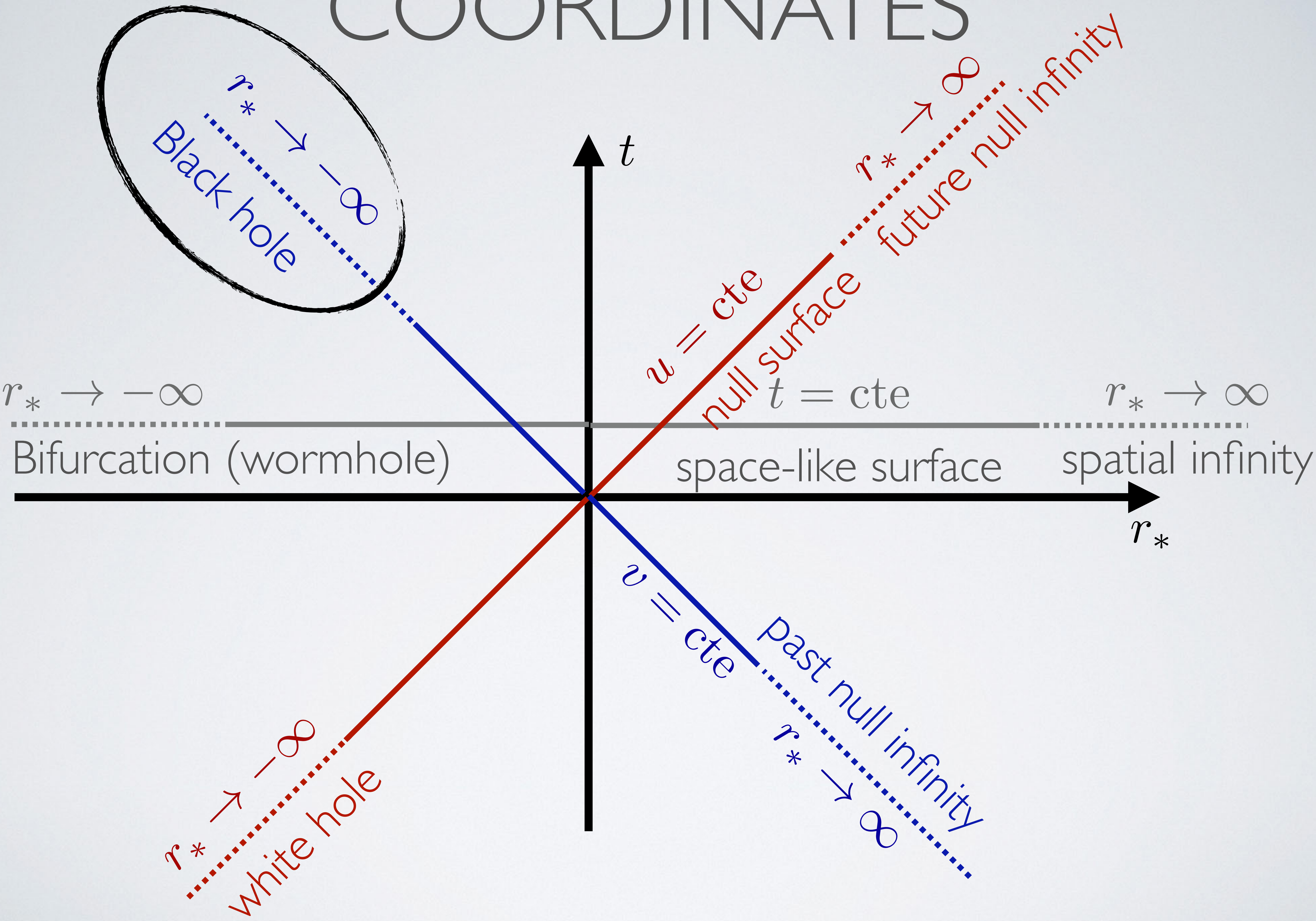

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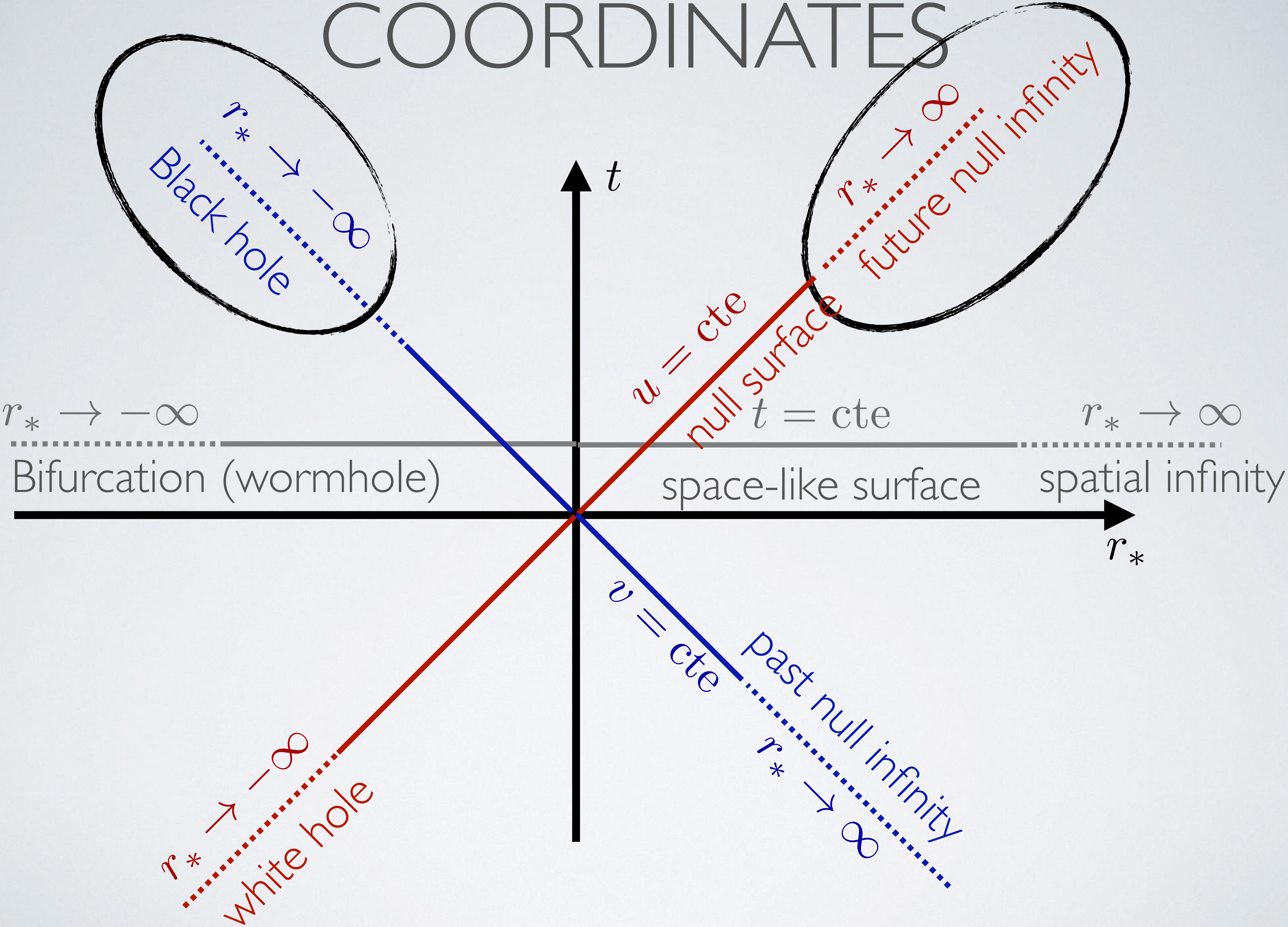

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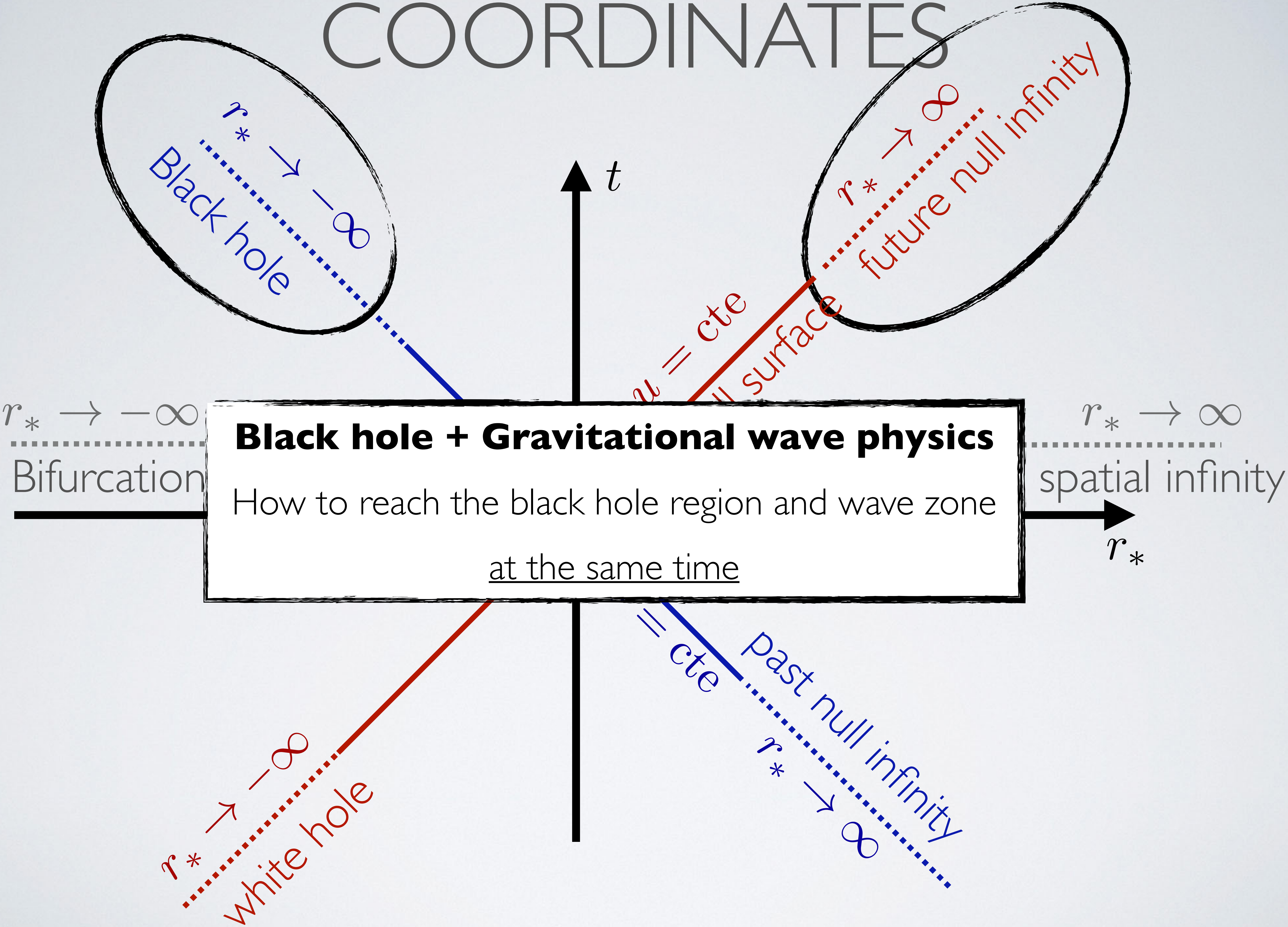
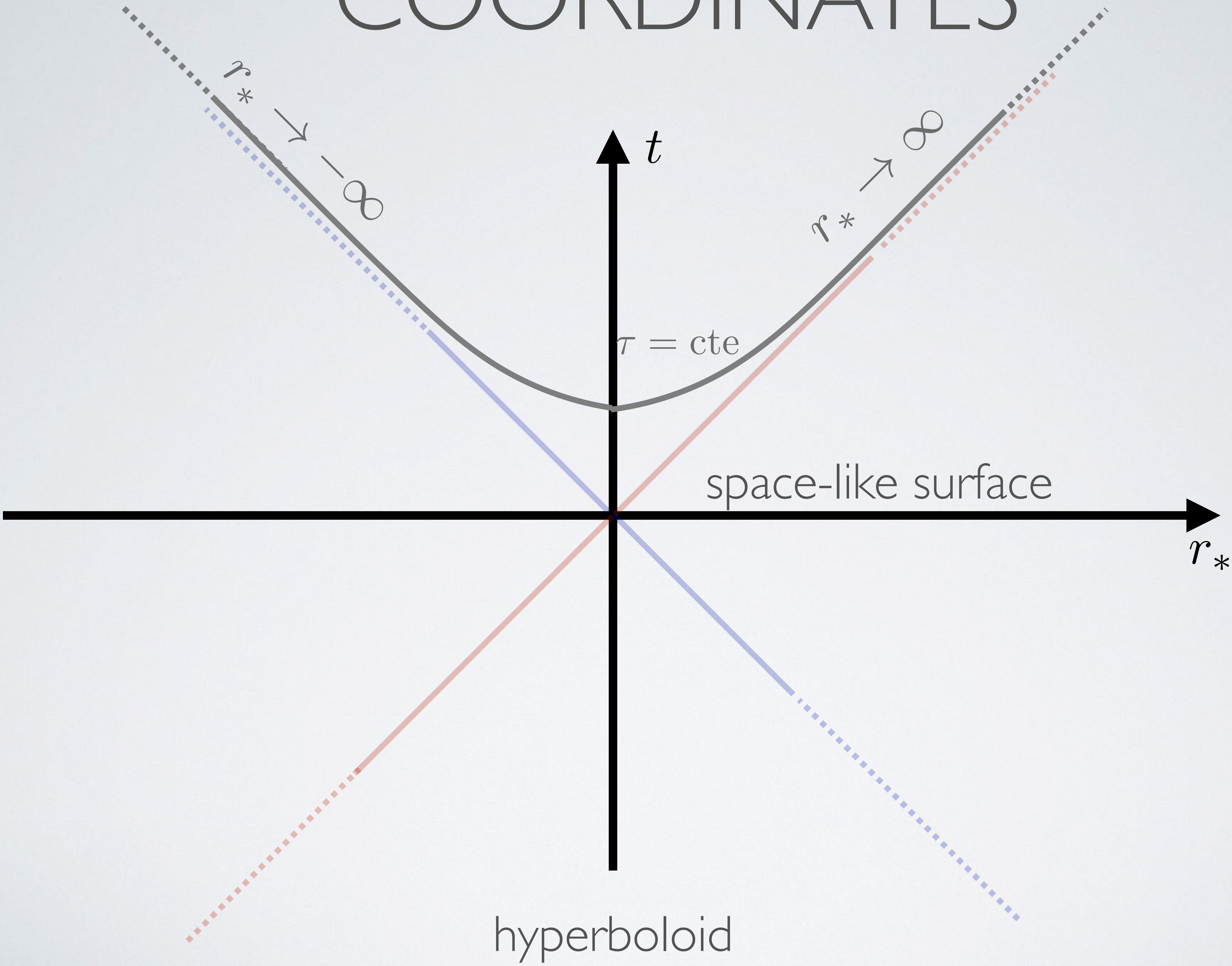


Black hole + Gravitational wave physics

How to reach the black hole region and wave zone

<u>at the same time</u>

COORDINATES



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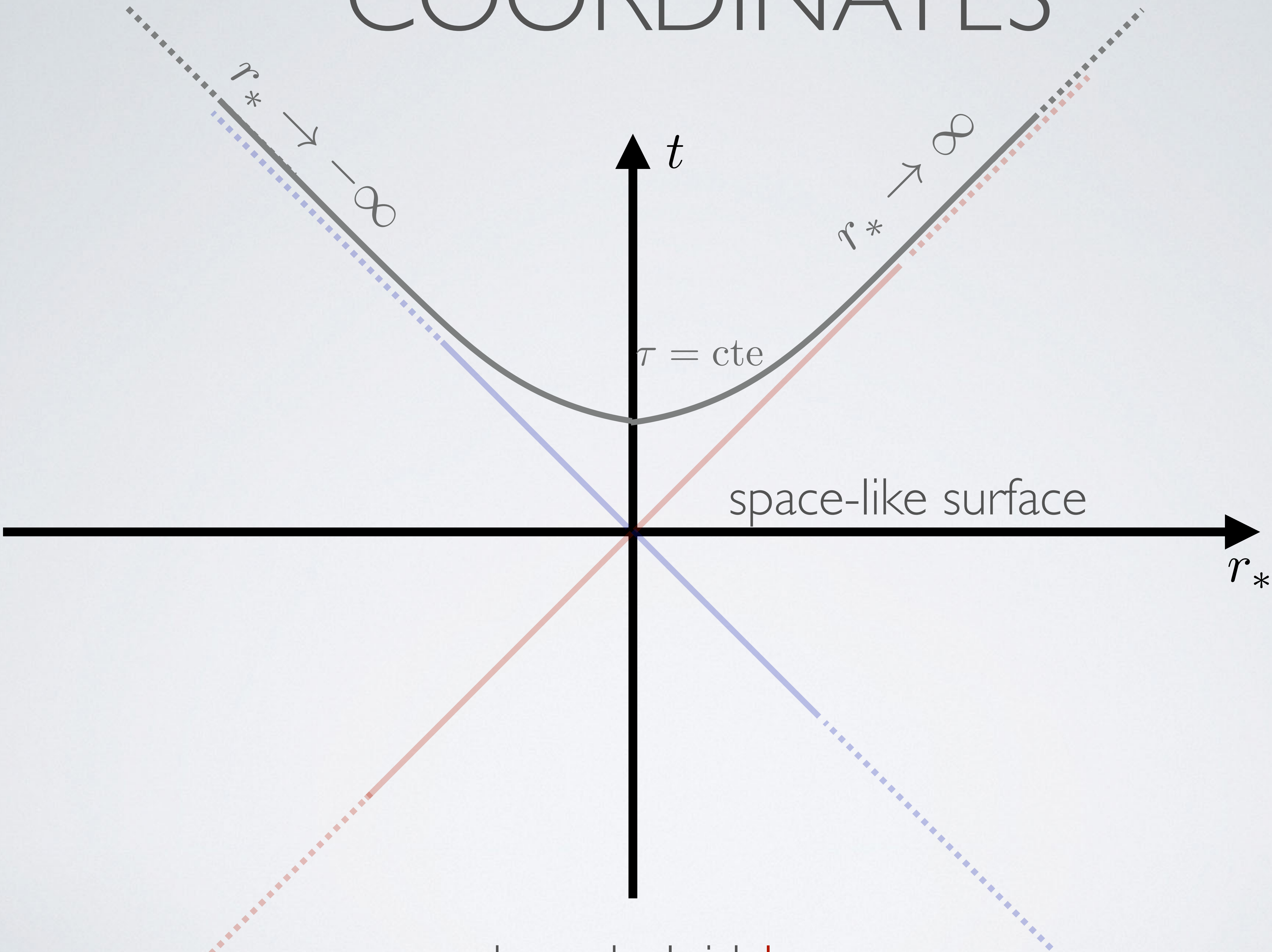
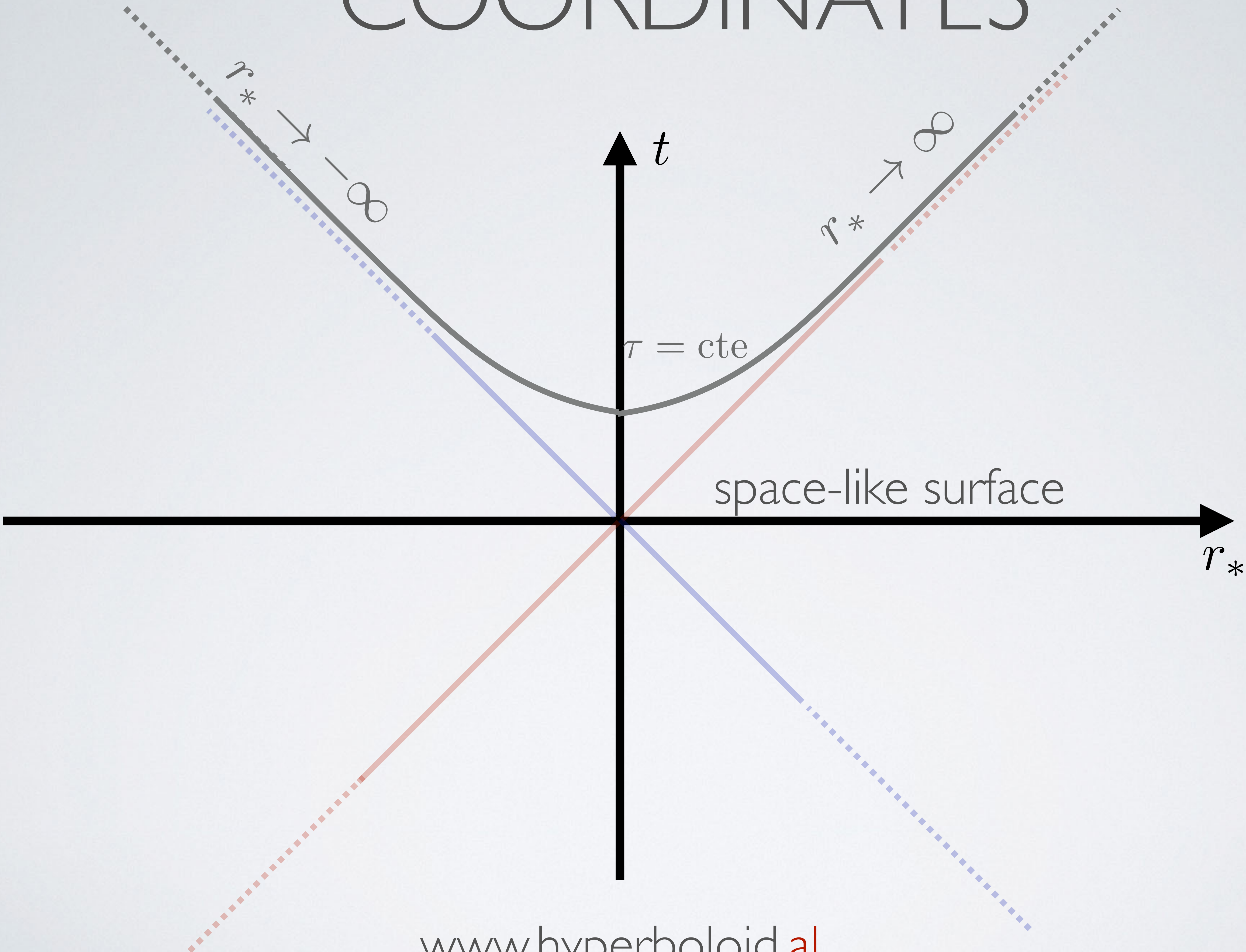


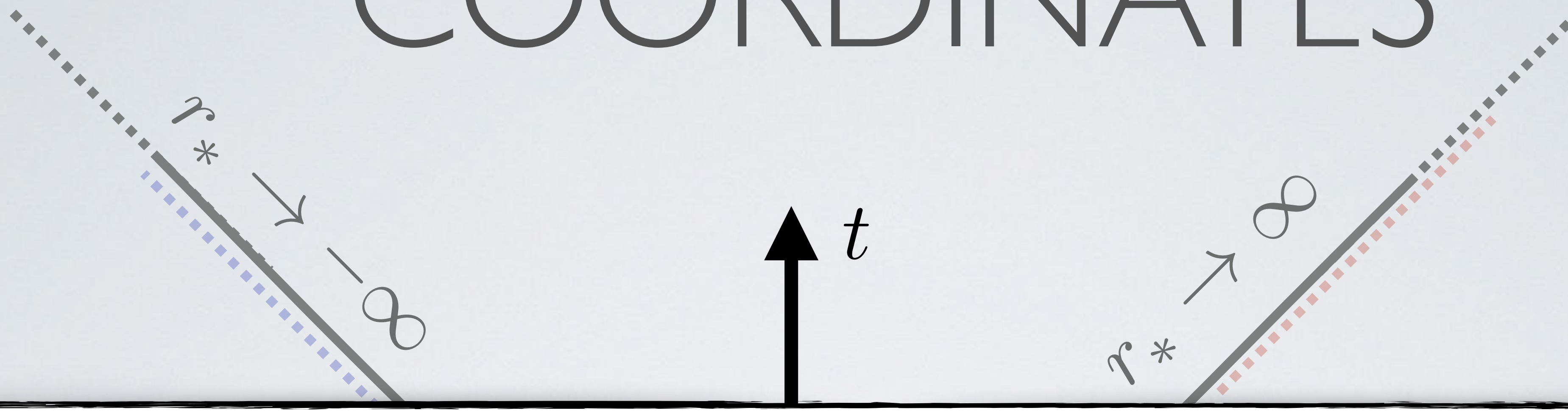
hyperboloidal

COORDINATES



www.hyperboloid.al

COORDINATES

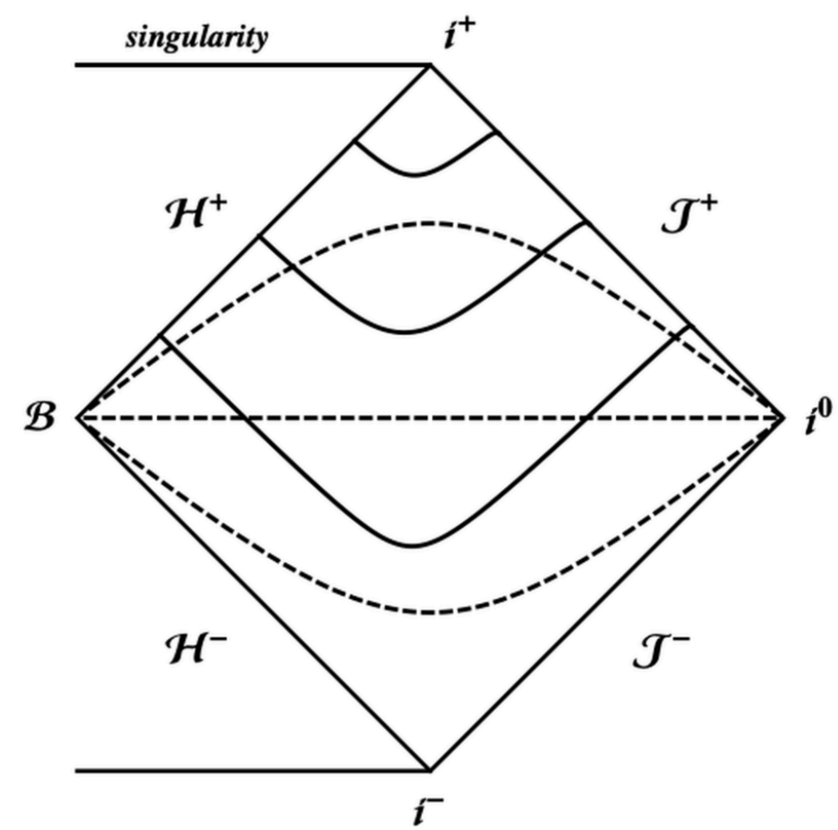


About People Events Blog Outreach

Hyperboloidal Research Network

Spacelike surfaces with asymptotically hyperbolic geometry are called **hyperboloidal**.

This site serves as a virtual center, bringing together a community of scientists who use hyperboloidal surfaces in their research.



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About People Events Blog Outreach

Events

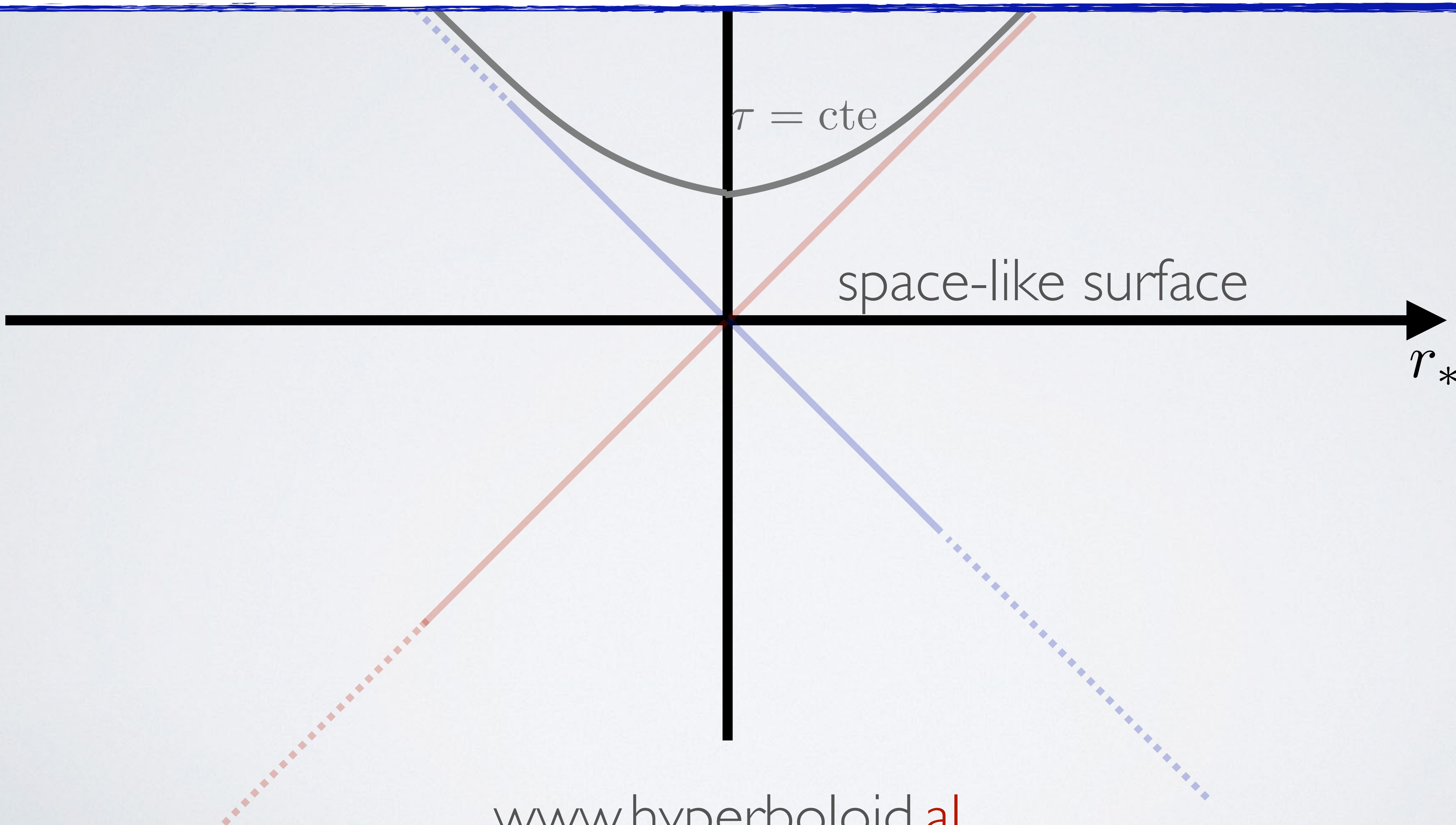
2026

Hyperboloidal Horizons 2026

Workshop of the hyperboloidal research network

Mon, 12 Jan 2026 — Fri, 16 Jan 2026 · Erwin Schrödinger Institute

David Hilditch, **Rodrigo Panosso Macedo**, **Alex Vañó-Viñuales**, **Anıl Zenginoğlu**



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About People Events Blog Outreach

Newsletter

Hyperbolic Times

We have a monthly newsletter, called Hyperbolic Times, that you can read and subscribe to here. The newsletter contains information about upcoming events, recent publications, and other news related to research in asymptotia.

Virtual Seminars

Hyperbolic Times

As a thematically focused but geographically distributed network, we hold virtual events via Zoom to exchange ideas. Our monthly virtual seminars are recorded. The first virtual seminar of the series will take place in October on Friday the 13th, 2023, at 14:00 GMT. Rodrigo Panosso Macedo will discuss the construction of minimal gauges in various spacetimes based on recent work.

YouTube Channel

We have a YouTube channel that will feature videos from our events.

HYPERBOLOIDAL GEOMETRY

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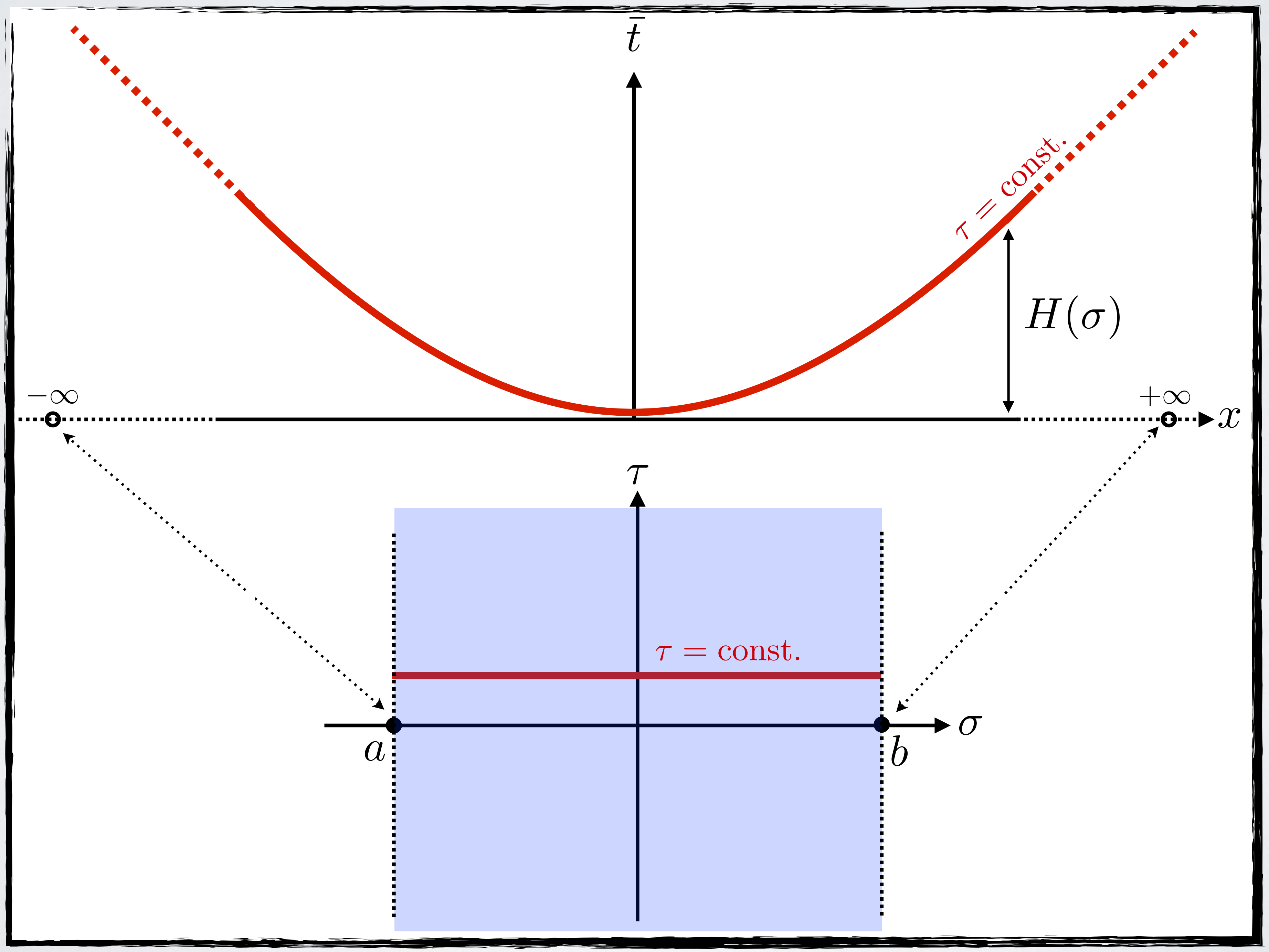
- Compactified hyperboloidal coordinates: $(\tau, \sigma, \theta, \varphi)$

HYPERBOLOIDAL GEOMETRY

- Compactified hyperboloidal coordinates: $(\tau, \sigma, \theta, \varphi)$

$$t = \lambda \left(\tau - H(\sigma) \right) \qquad r = \lambda \frac{\rho(\sigma)}{\sigma} \Leftrightarrow r_* = \lambda\, x(\sigma)$$



PERTURBATION THEORY

- **Wave equation on black-hole background**

$$-\Psi_{,tt} + \Psi_{,r_* r_*} - \mathcal{V}(r)\,\Psi = 0$$

$\Psi(t,r)$: Perturbation field

$\mathcal{V}(r)$: Potential

- **Hyperboloidal coordinates**

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$$-\Psi_{\tau\tau} + L_1\Psi + L_2\Psi_{,\tau} = 0$$

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(Singular) Sturm-Liouvielle operator

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L_1: (Singular) Sturm-Liouvielle operator

L_2: Encodes energy dissipation

PERTURBATION THEORY

- **Wave equation on black-hole background**

$\Psi(t,r)$

$\mathcal{V}(r)$: F

- **Hyp**

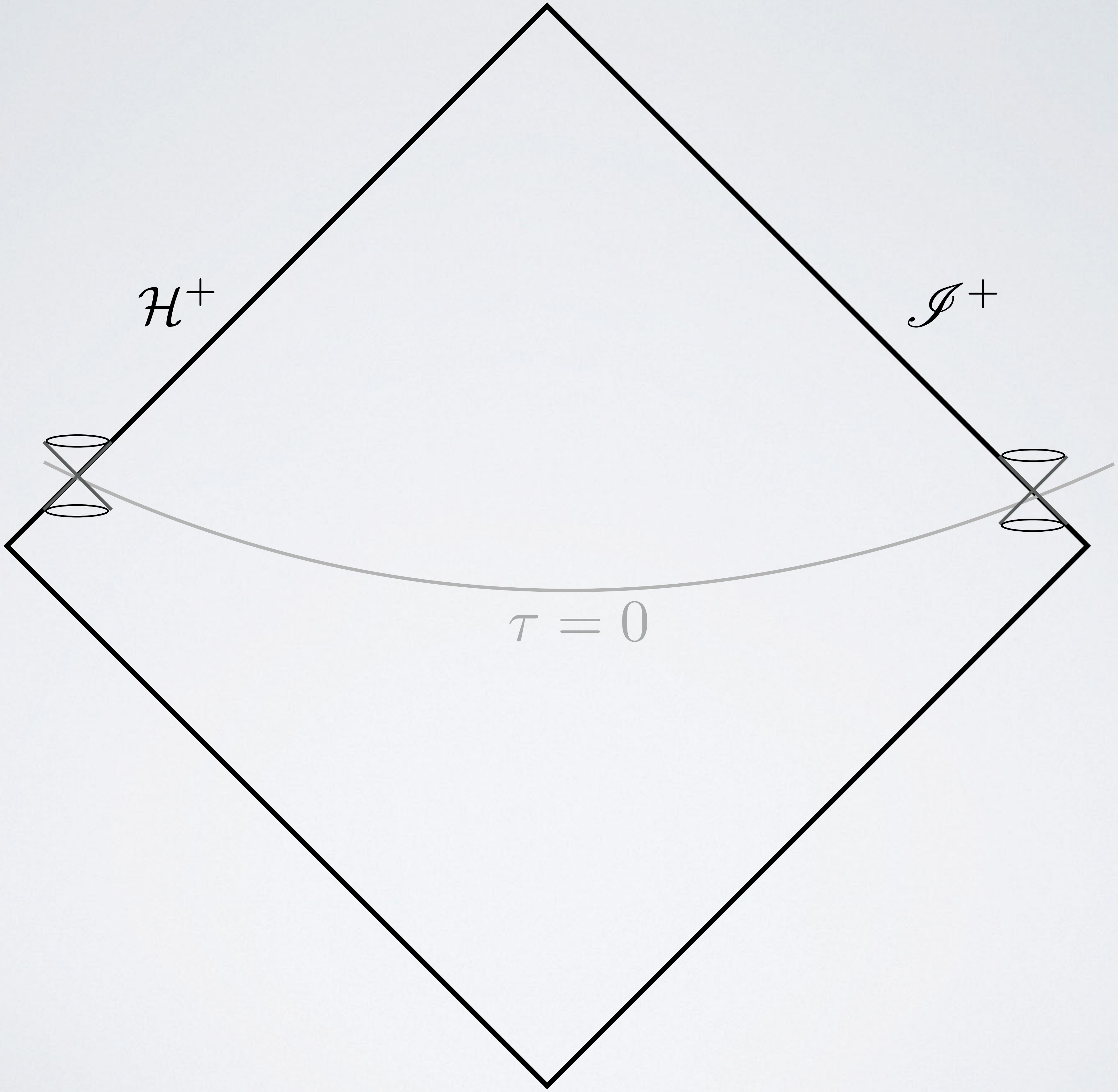
(Singular)

Boundary Conditions X Regularity

Ingoing/Outgoing external boundary conditions in the original problem are now geometrically taken into account via hyperboloidal slices.

At the differential equation level, the physical scenario (Black Hole+Wave Zone) is described by the equations' regular solutions

TIME DOMAIN



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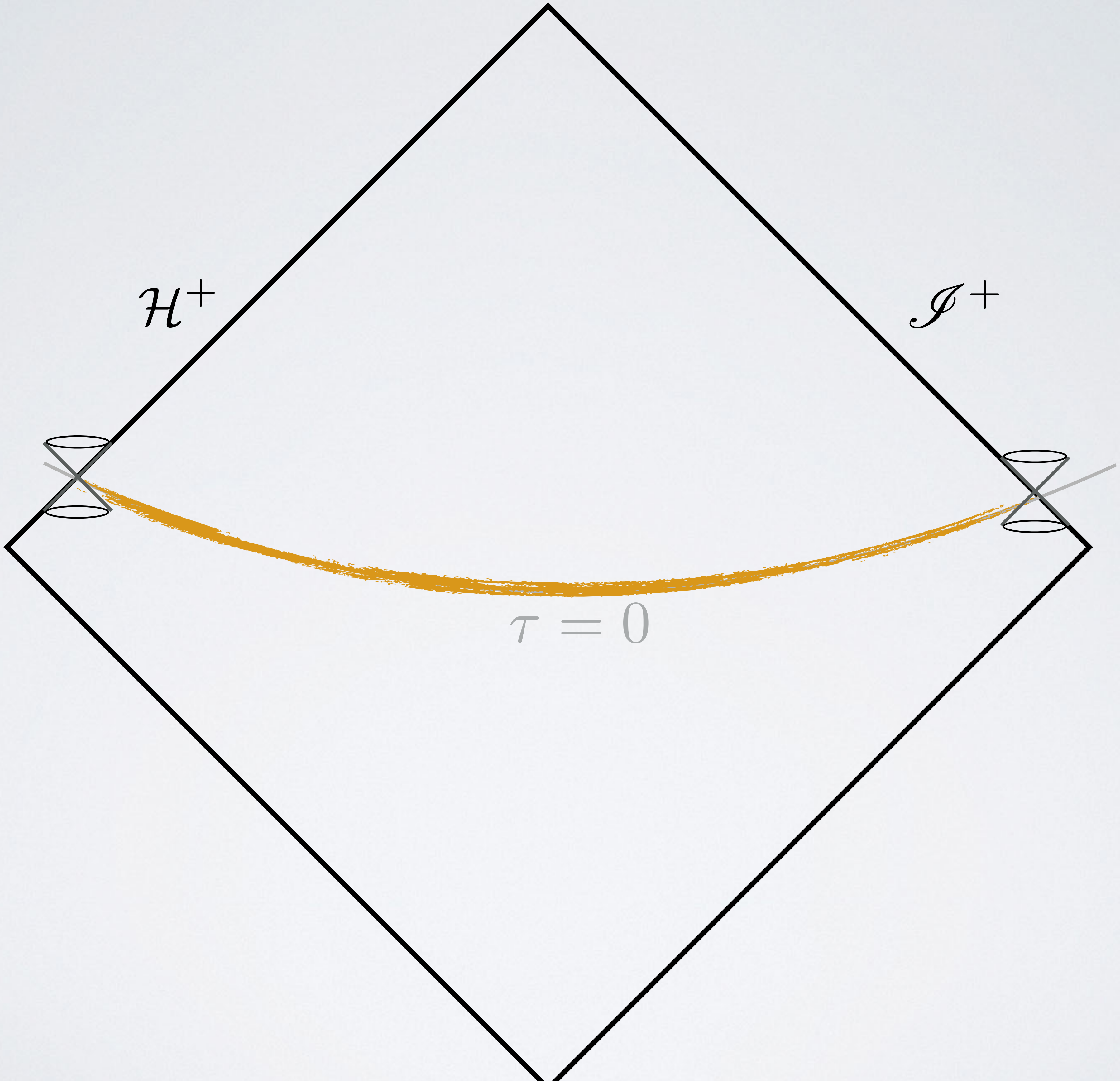

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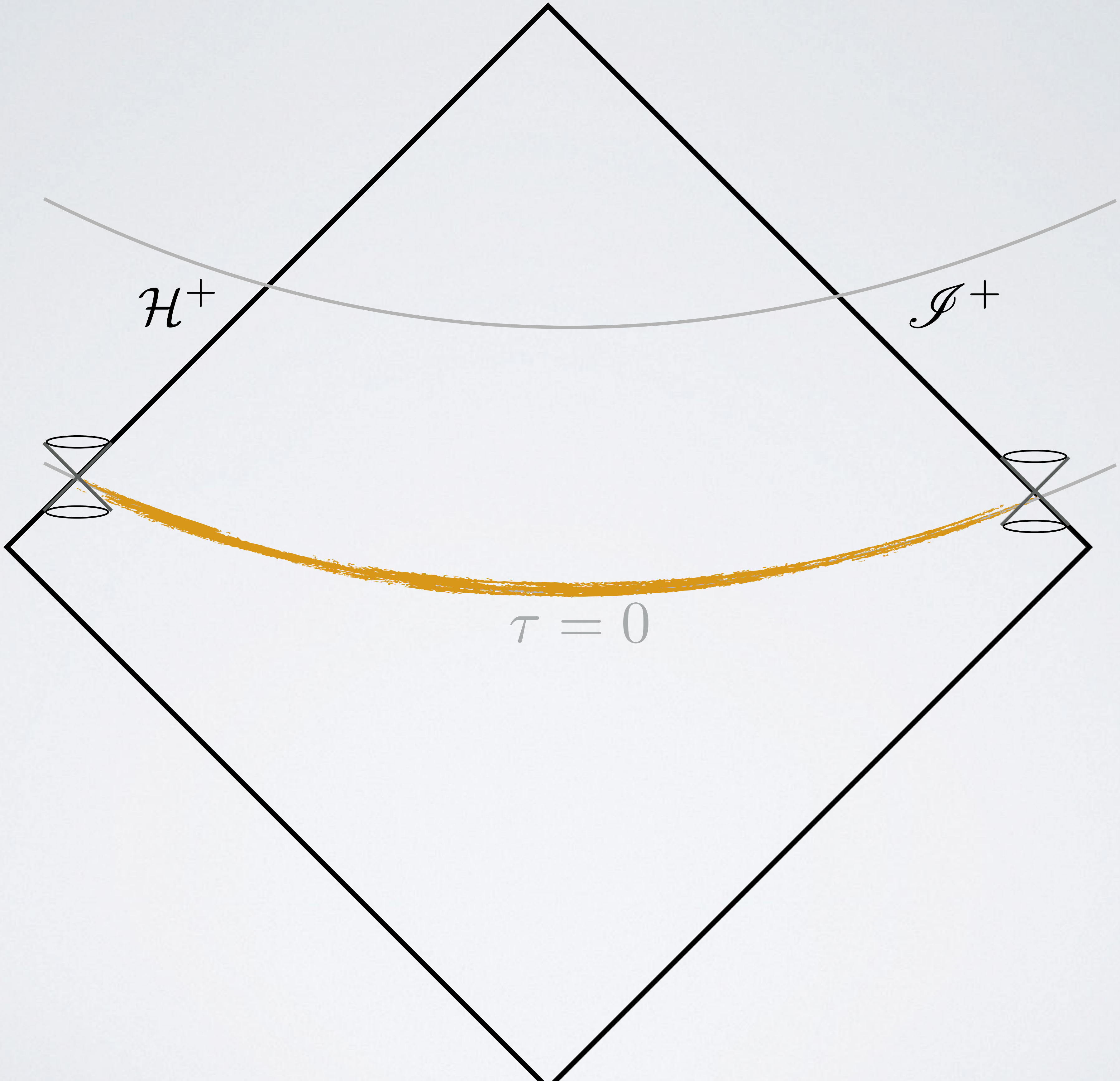

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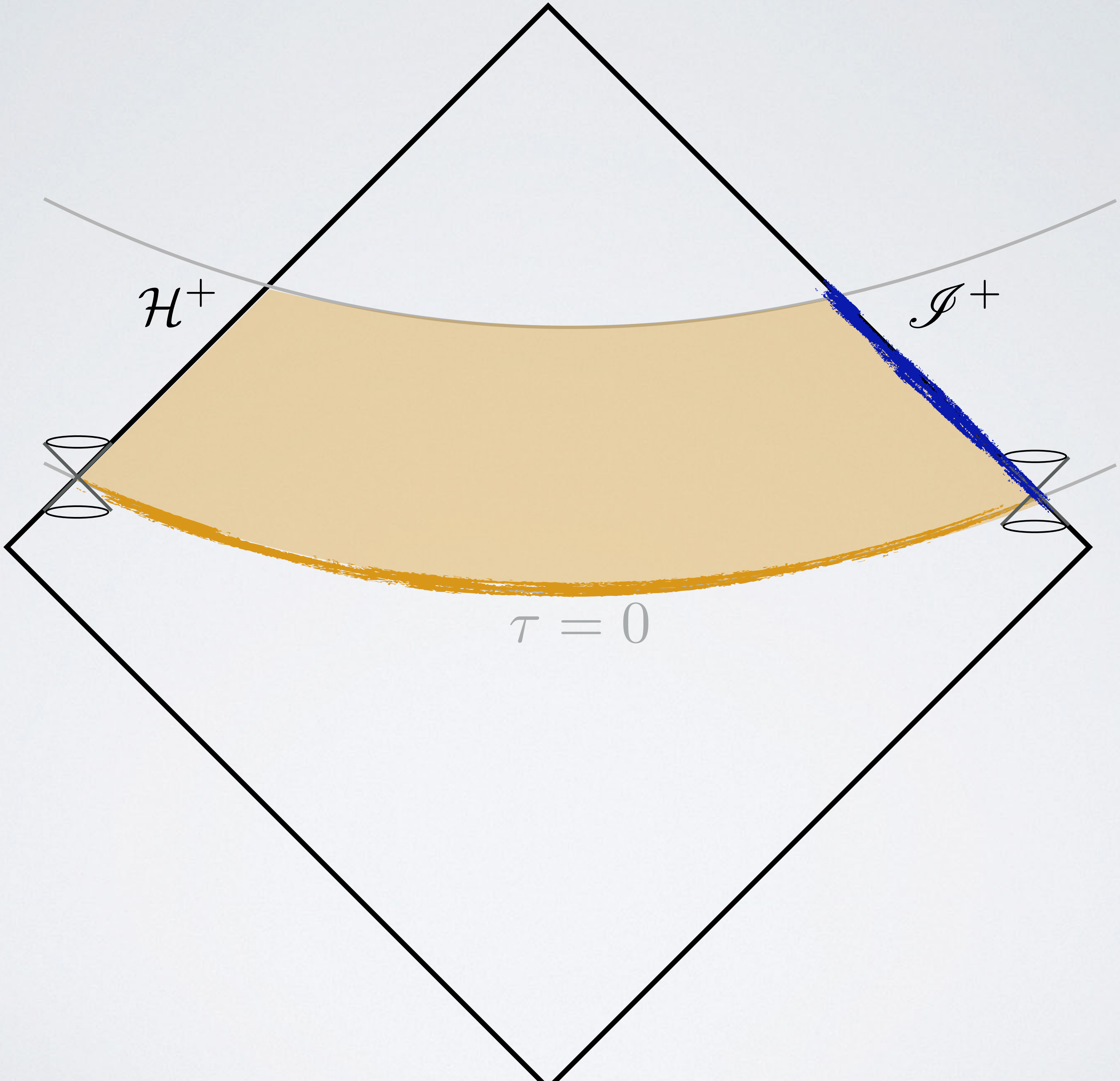

FREQUENCY DOMAIN

- Wave Equation:

FREQUENCY DOMAIN

- Wave Equation: $-\Psi_{\tau\tau} + L_1 \Psi + L_2 \Psi_{,\tau} = 0$

FREQUENCY DOMAIN

- Wave Equation: $-\Psi_{\tau\tau} + L_1\Psi + L_2\Psi_{,\tau} = 0$

$$\downarrow \quad \Phi = \partial_\tau \Psi \quad , \quad u = \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}$$

$$\partial_\tau u = iLu$$

$$L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)$$

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- Wave Equation: $-\Psi_{\tau\tau} + L_1\Psi + L_2\Psi_{,\tau} = 0$

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Eigenvalue Problem

Fourier (or Laplace) Transform: $u \sim e^{i\omega\tau}\hat{u}$

$$L\hat{u} = \omega\hat{u}$$

FREQUENCY DOMAIN

- Wave Equation:

New tools in Gravity/Optics

- Import tools from theory of non-self adjoint operators

e.g. pseudospectrum [see J.L. Jaramillo, R.P. Macedo, L. Sheik PRX 2022]

- Hyperboloidal Framework in optical system

Lamis Al Sheik PhD Thesis

- See Jeremy Besson Talk on Wed

Fourier (or Laplace) Transform:

$$L\hat{u} = \omega\hat{u}$$

NORMAL OPERATORS: SPECTRAL THEOREMS

Def. *Given matrix* L *and its adjoint* $L^\dagger$. *Then* L *is* <u>*normal*</u> *iff* $[L, L^\dagger] = 0$

Ex. *Symmetric, hermitian, orthogonal, unitary…*

Spectral Theo. (matrices): L *is normal iff is unitarily diagonalisable*

Obs. *Theorem extends to normal operators in Hilbert (or Banach) space*

↓

Hermitian Physics (self adjoint operators):

Eigenvectors are orthogonal and form complete basis

Eigenvalues are stable $L \to L + \epsilon\delta L \Rightarrow \lambda \to \lambda + \epsilon\delta\lambda$

NON-NORMAL OPERATORS: NO SPECTRAL THEOREMS

If $[L, L^\dagger] \neq 0$

Completeness more difficult to study

Eigenvectors not necessarily orthogonal

Spectral Instabilities

Non-Hermitian Physics (non-self adjoint operator):

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$$L(\epsilon) = L + \epsilon \delta L, \quad |\delta L| = 1$$

$$|\lambda_i(\epsilon) - \lambda_i| \leq \kappa_i \, \epsilon$$

Eigenvalue Condition Number: $\kappa_i = \dfrac{||r_i|| \, ||l_i||}{|r_i \cdot l_i|}$

$L \, r_i = \lambda_i \, r_i$ (Right eigenvector)

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<u>*Small perturbation in operator; small displacement in eigenvalue*</u>

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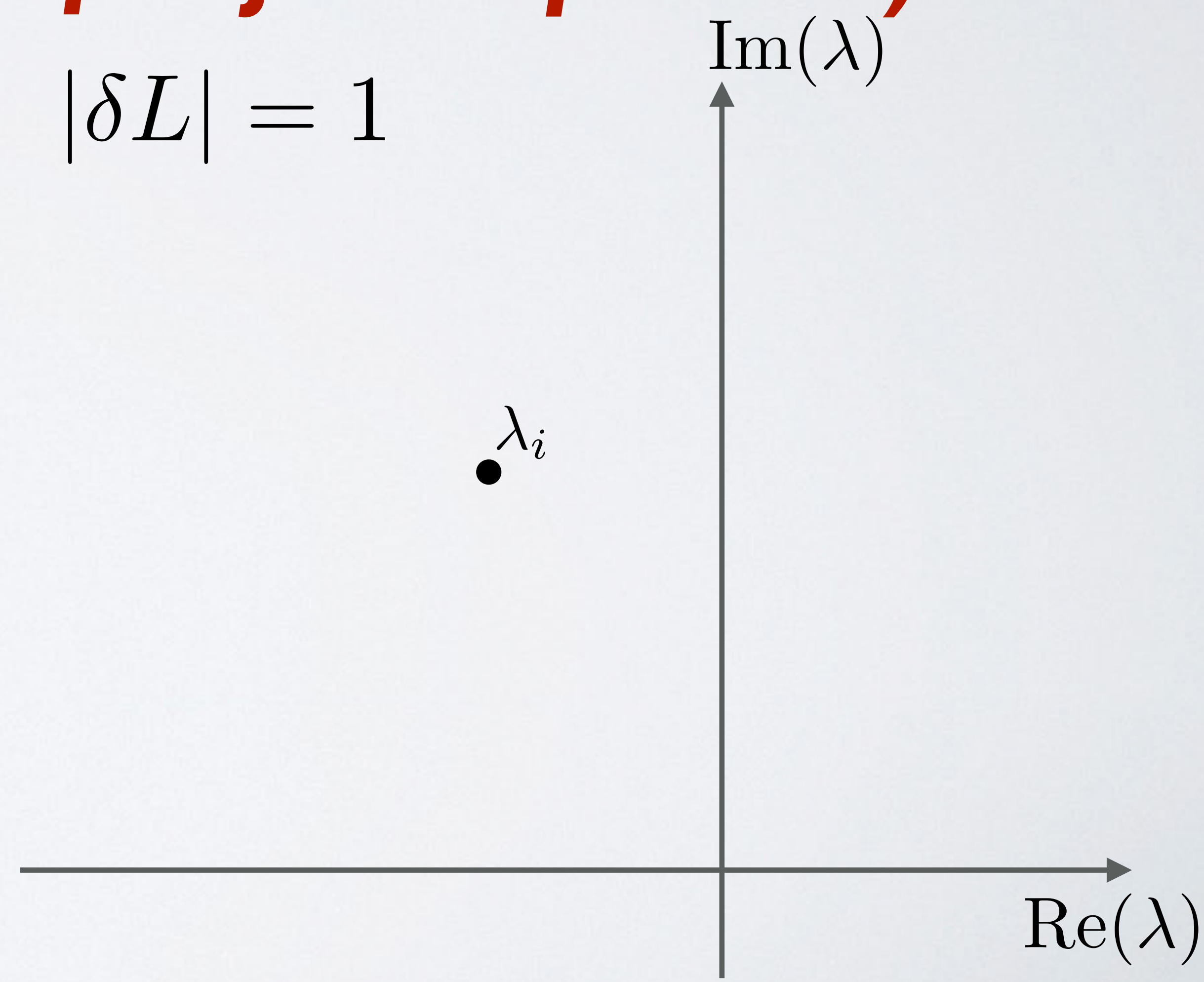

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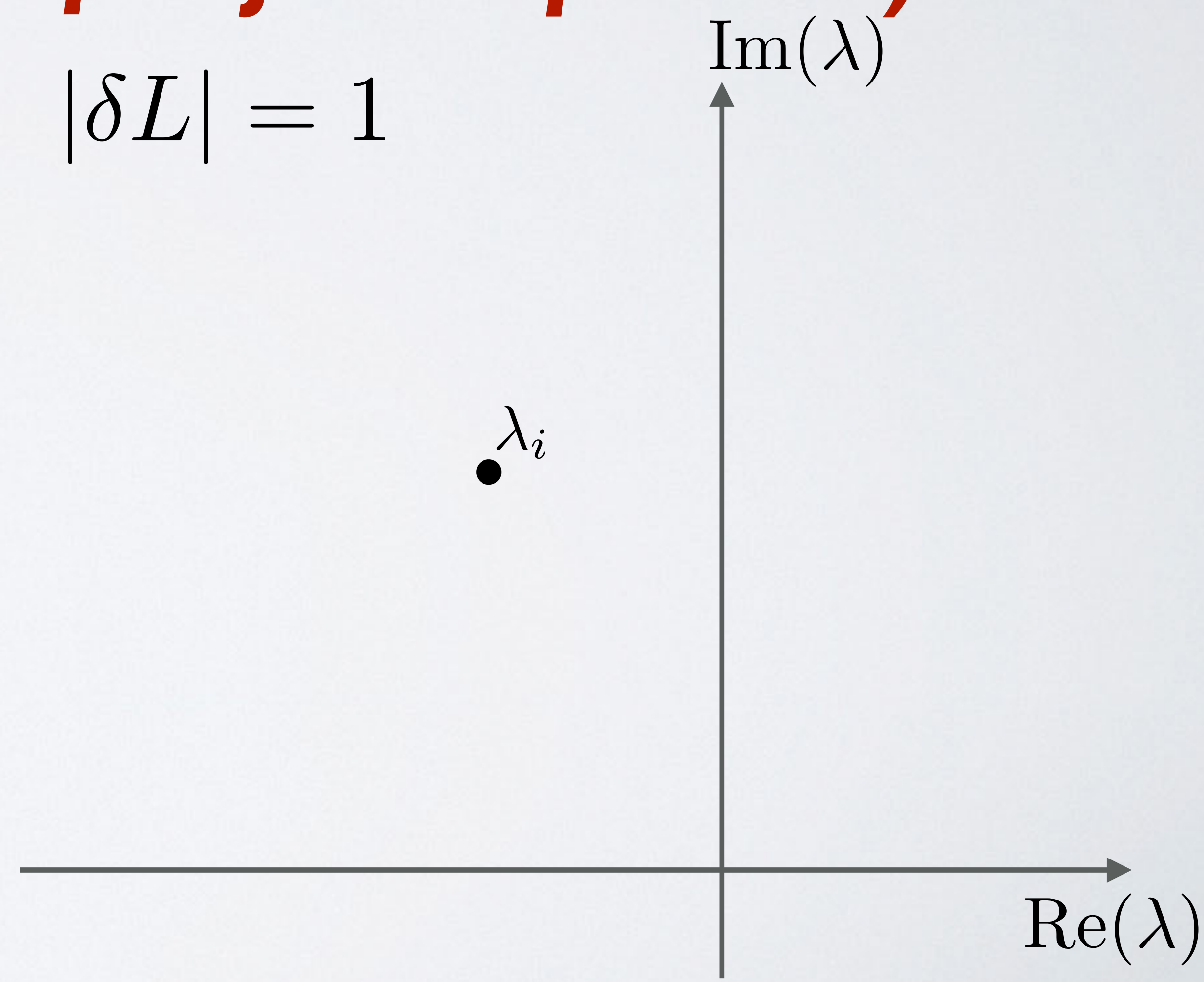

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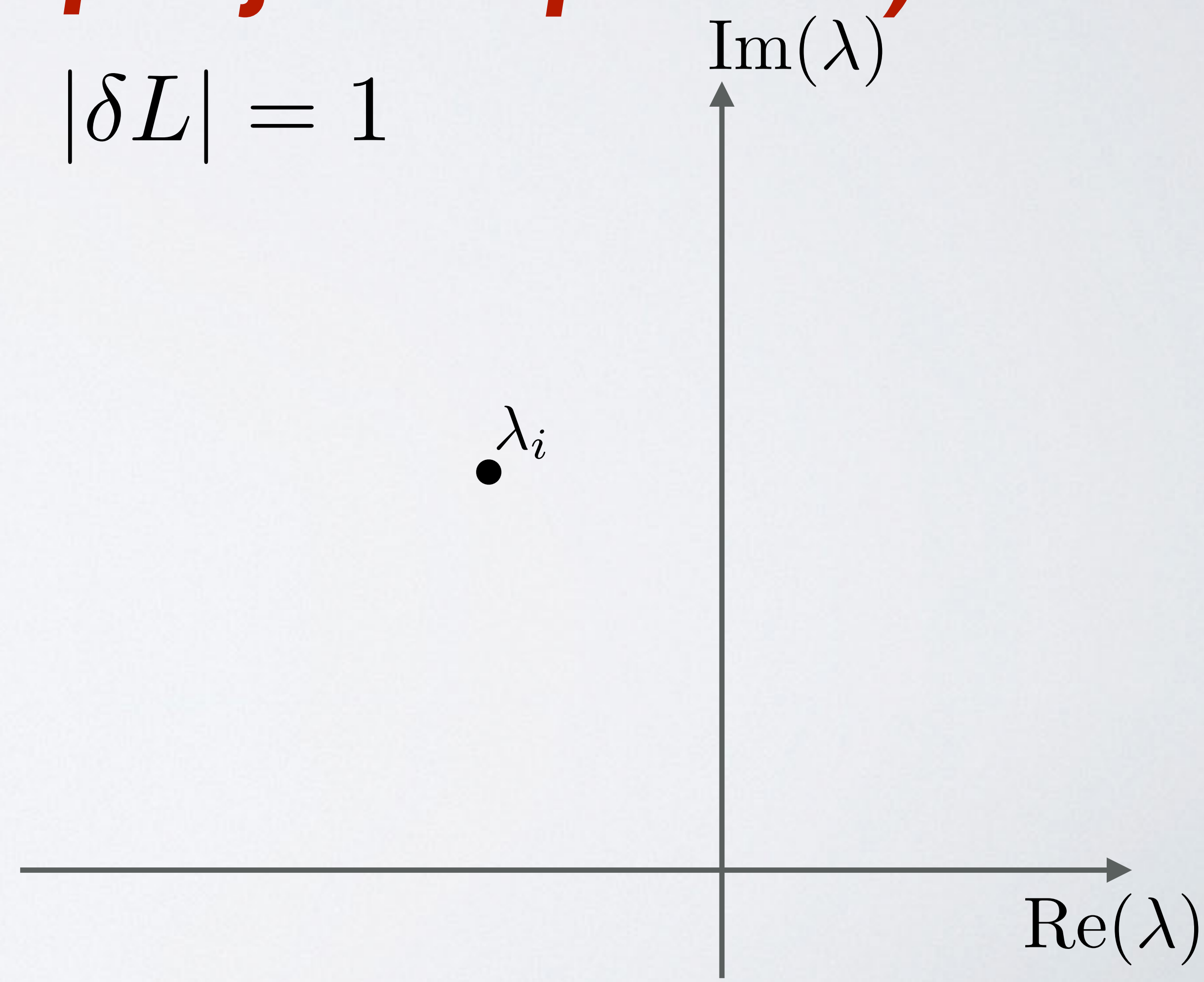

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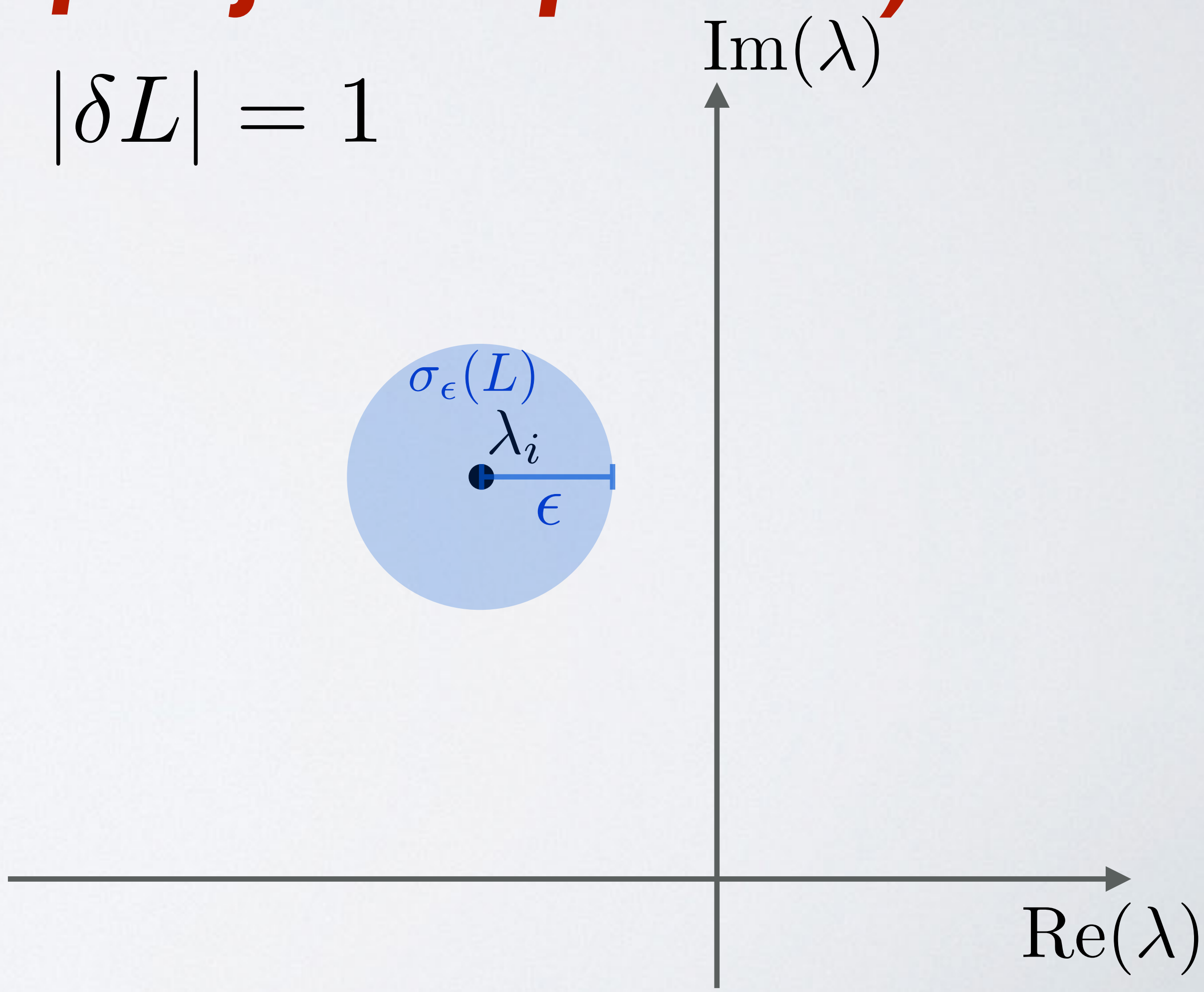

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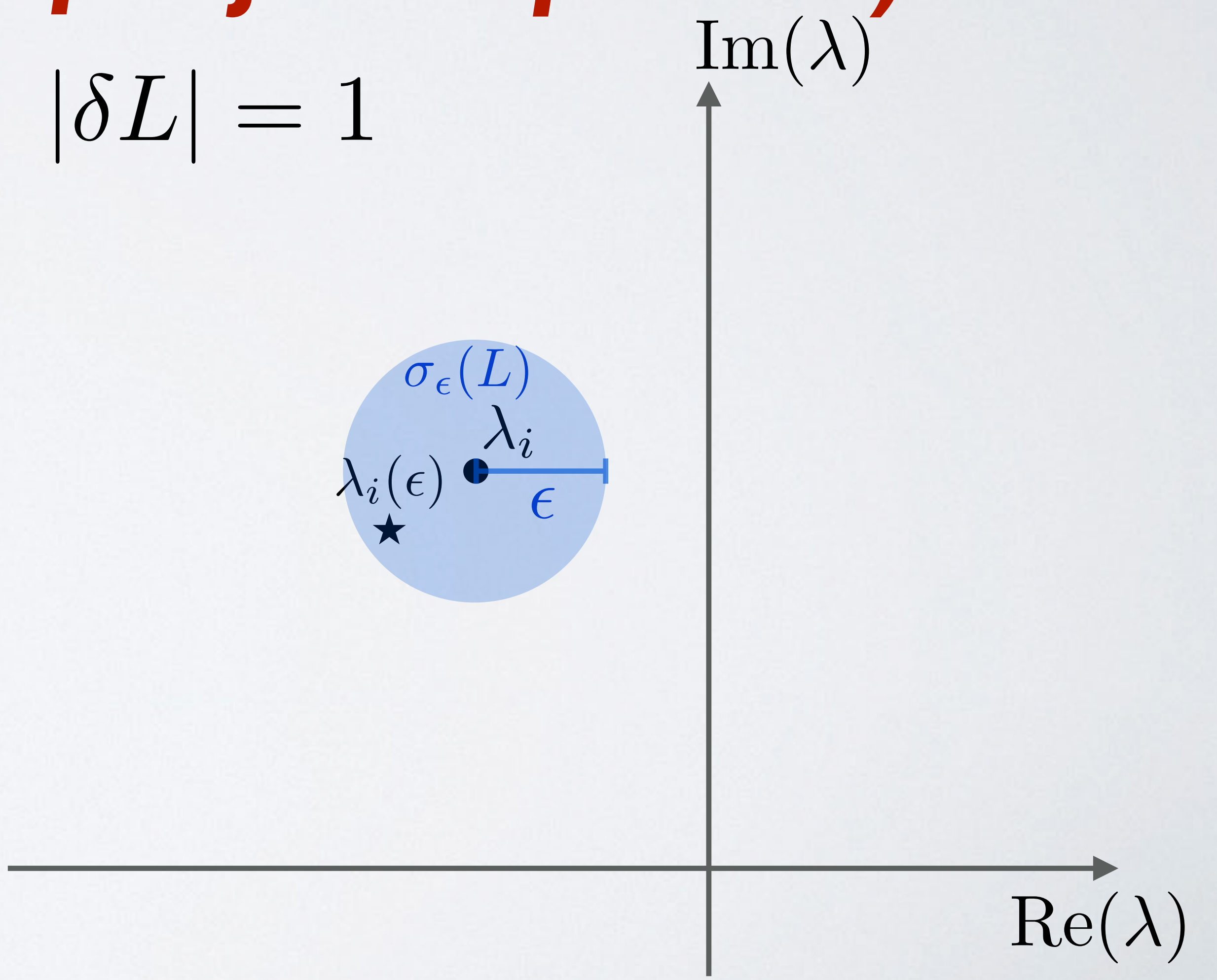

NON… TORS:
NO… EMS

If $[L, L^\dagger] \neq$ …

Non-Her… erator):

$\epsilon < 1$ …

Spectra:

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Pseudospec…

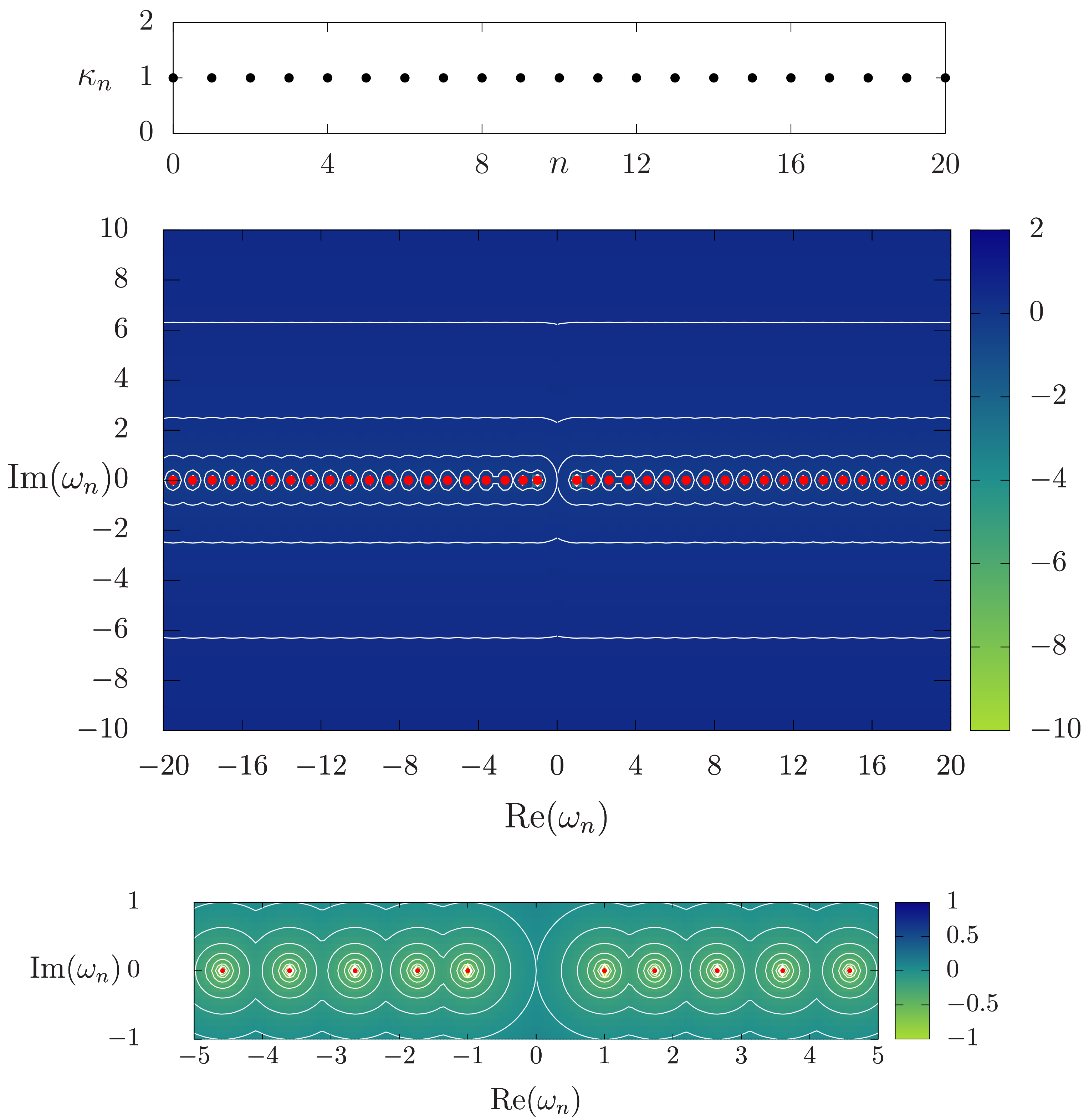
$\sigma_\epsilon(L) : ||L$ …

self adjoint operator:

Im(λ)

Re(λ)

Wave on Sphere (self-adjoint problem)



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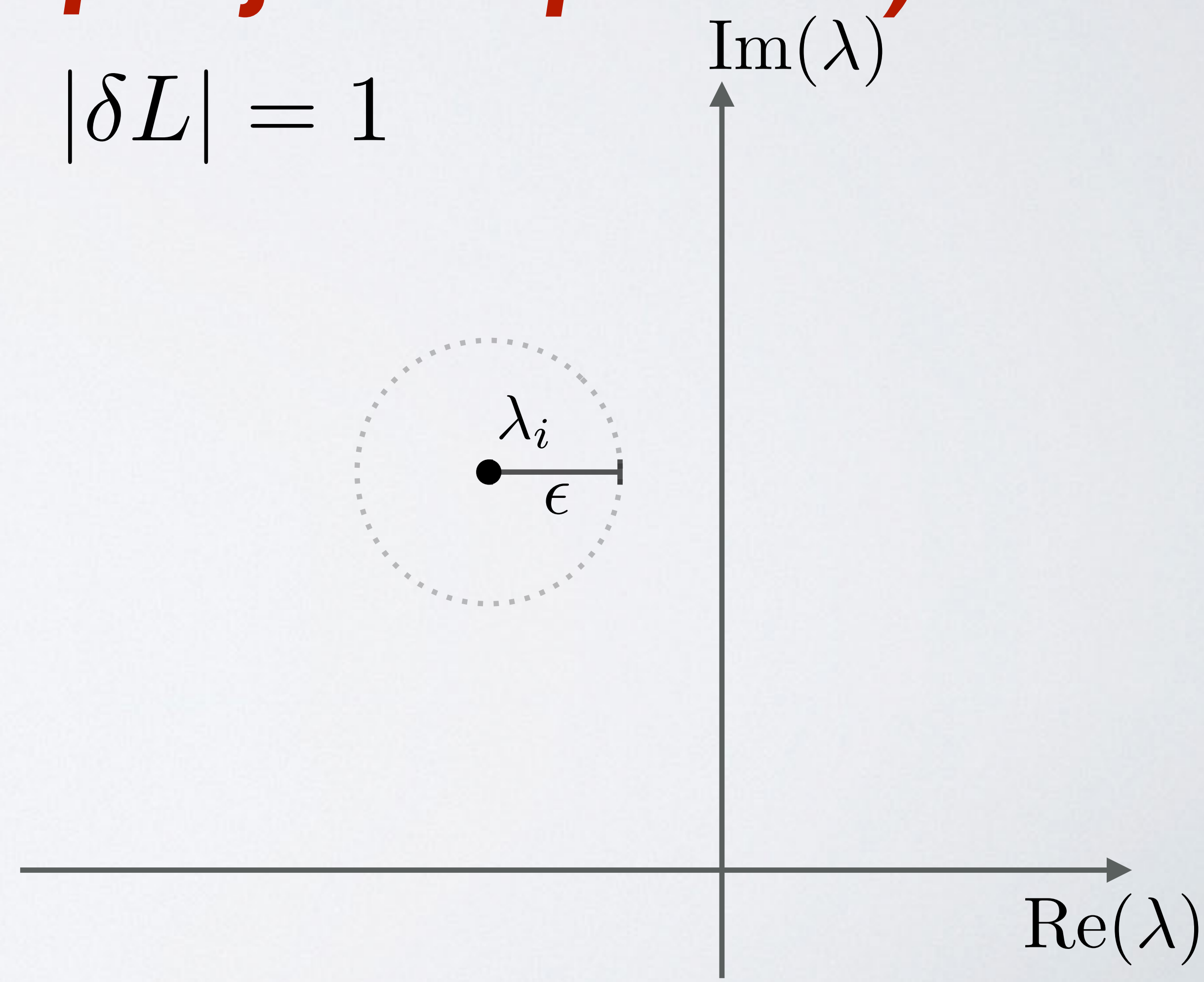

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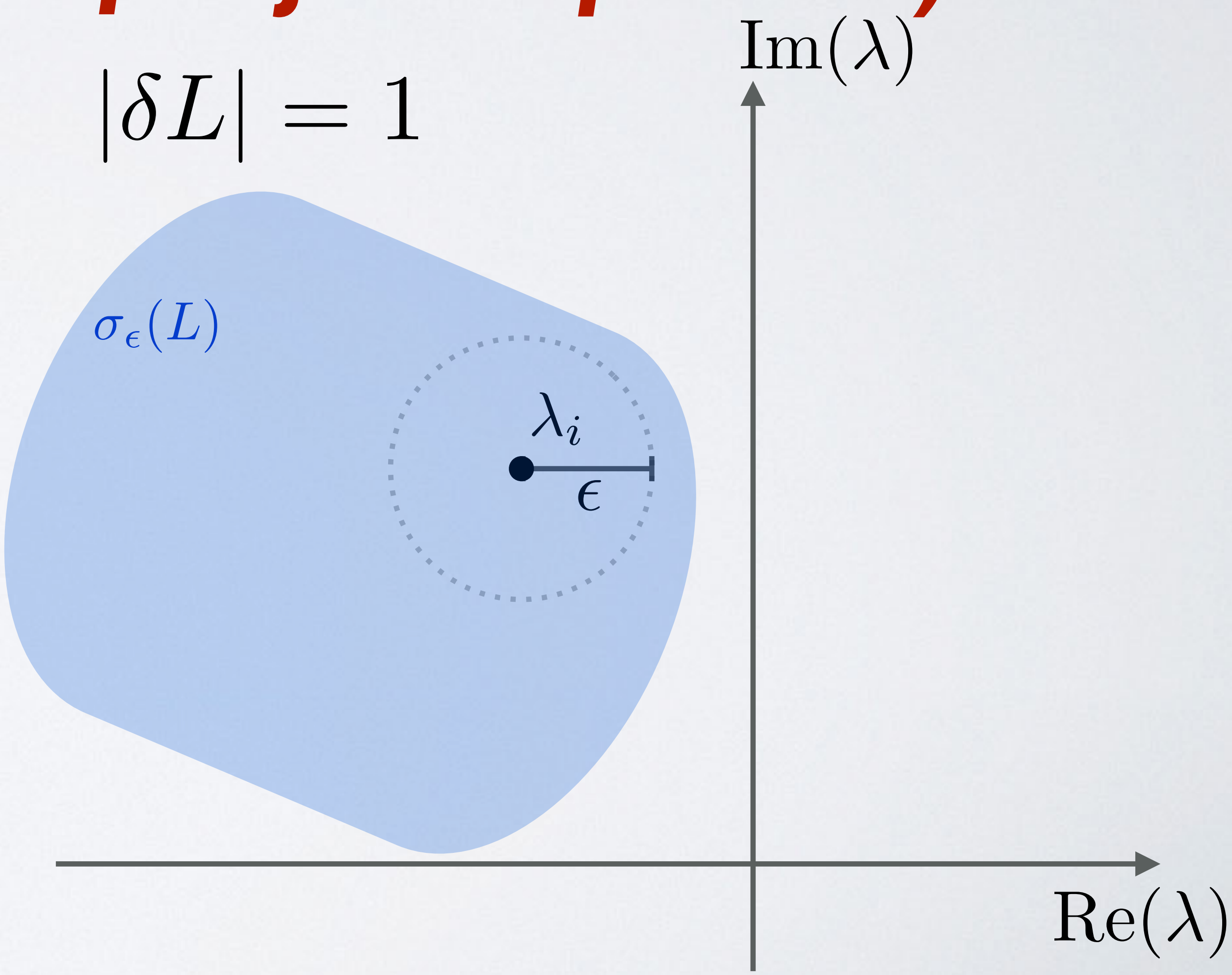


non-self adjoint operator:

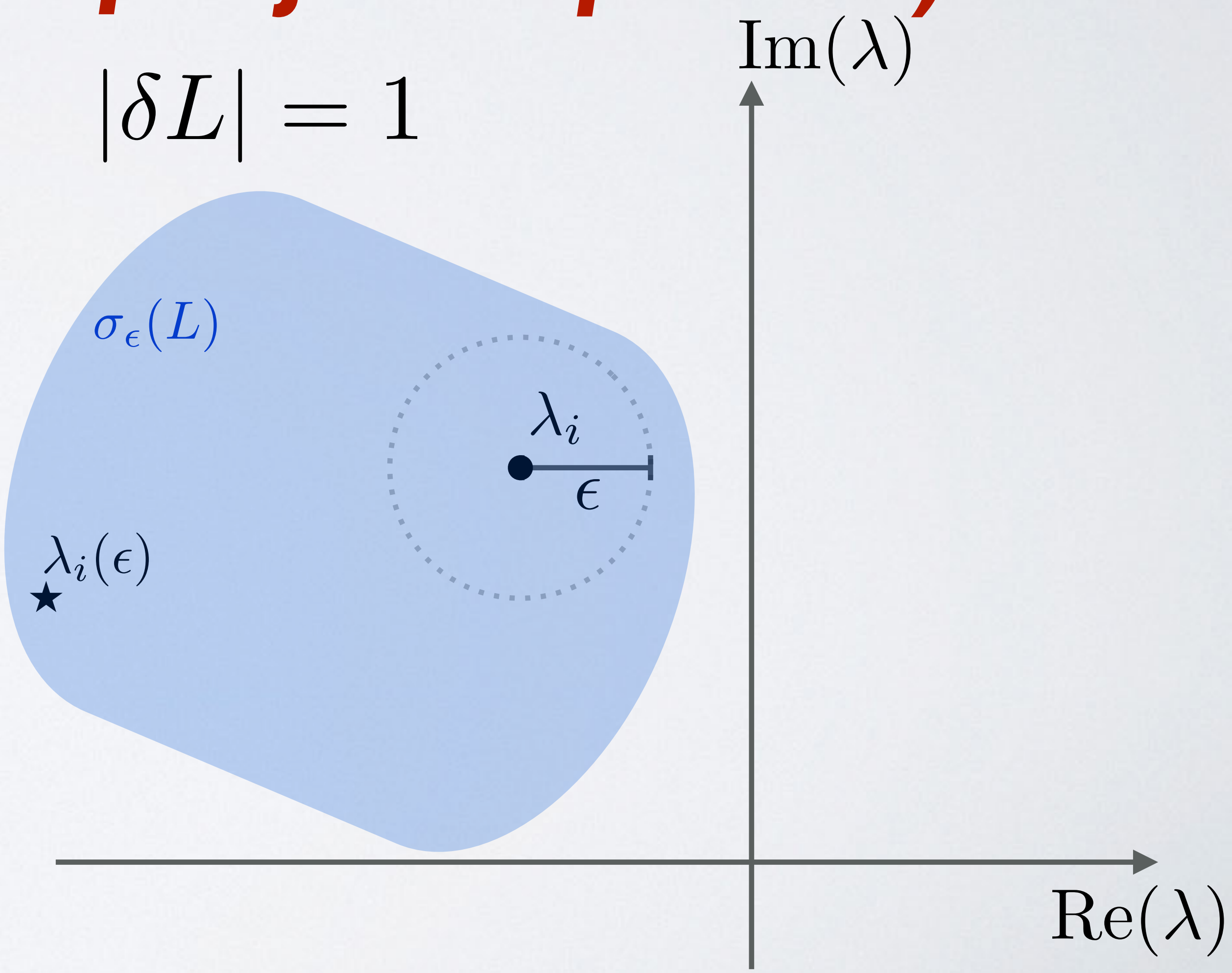
NON-NORMAL OPERATORS: NO SPECTRAL THEOREMS

If $[L, L^\dagger] \neq 0$

Completeness more difficult to study
Eigenvectors not necessarily orthogonal
Spectral Instabilities

Non-Hermitian Physics (non-self adjoint operator):

$$\epsilon < 1 \qquad L(\epsilon) = L + \epsilon\delta L, \quad |\delta L| = 1$$



Spectra:

$$\sigma(L) : ||L - \lambda_i \mathbb{I}|| = 0 \quad \text{(Eigenvalue)}$$

Pseudospectra:

$$\sigma_\epsilon(L) : ||L - \lambda_i \mathbb{I}|| < \epsilon$$

non-self adjoint operator:

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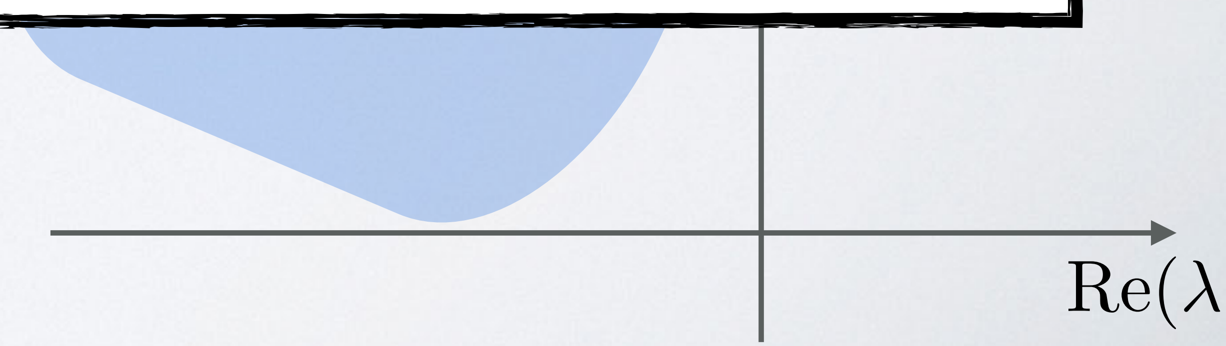
Take away message:

- The unperturbed operator contains all pieces of information to assess potential spectral instabilities
- Tools to measure: condition number and pseudospectra

$\sigma_\epsilon(L) : ||L - \lambda_i \mathbb{I}|| < \epsilon$



non-self adjoint operator:

SPECTRAL INSTABILITY

- Condition Number + Pseudospectral tools applied to Schwarzschild

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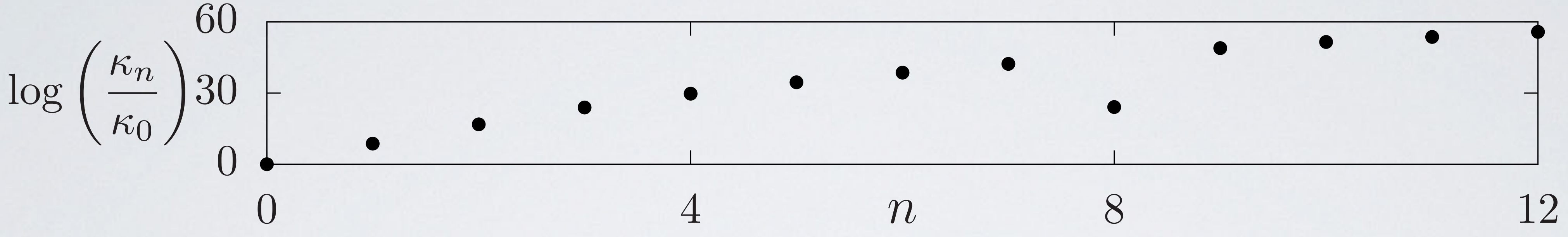

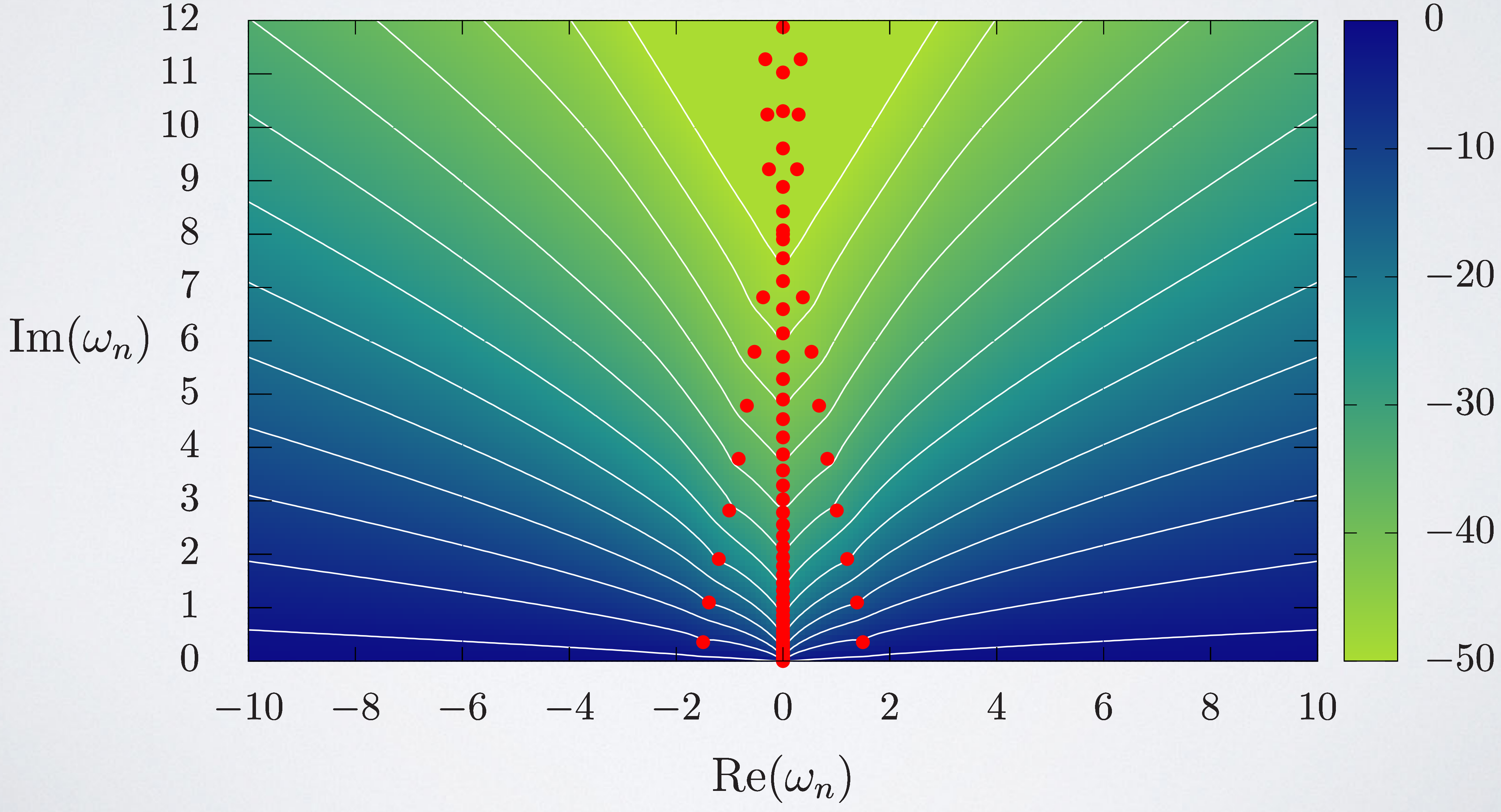

SPECTRAL INSTABILITY

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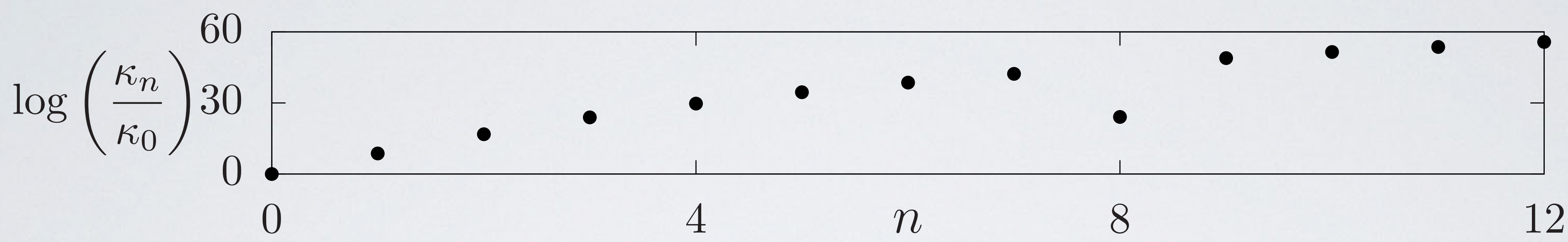
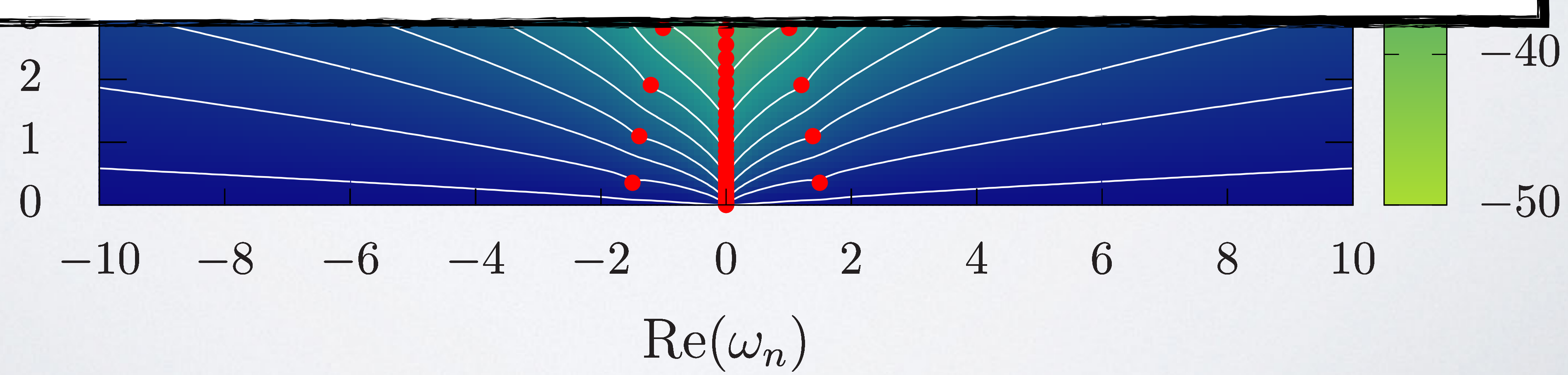


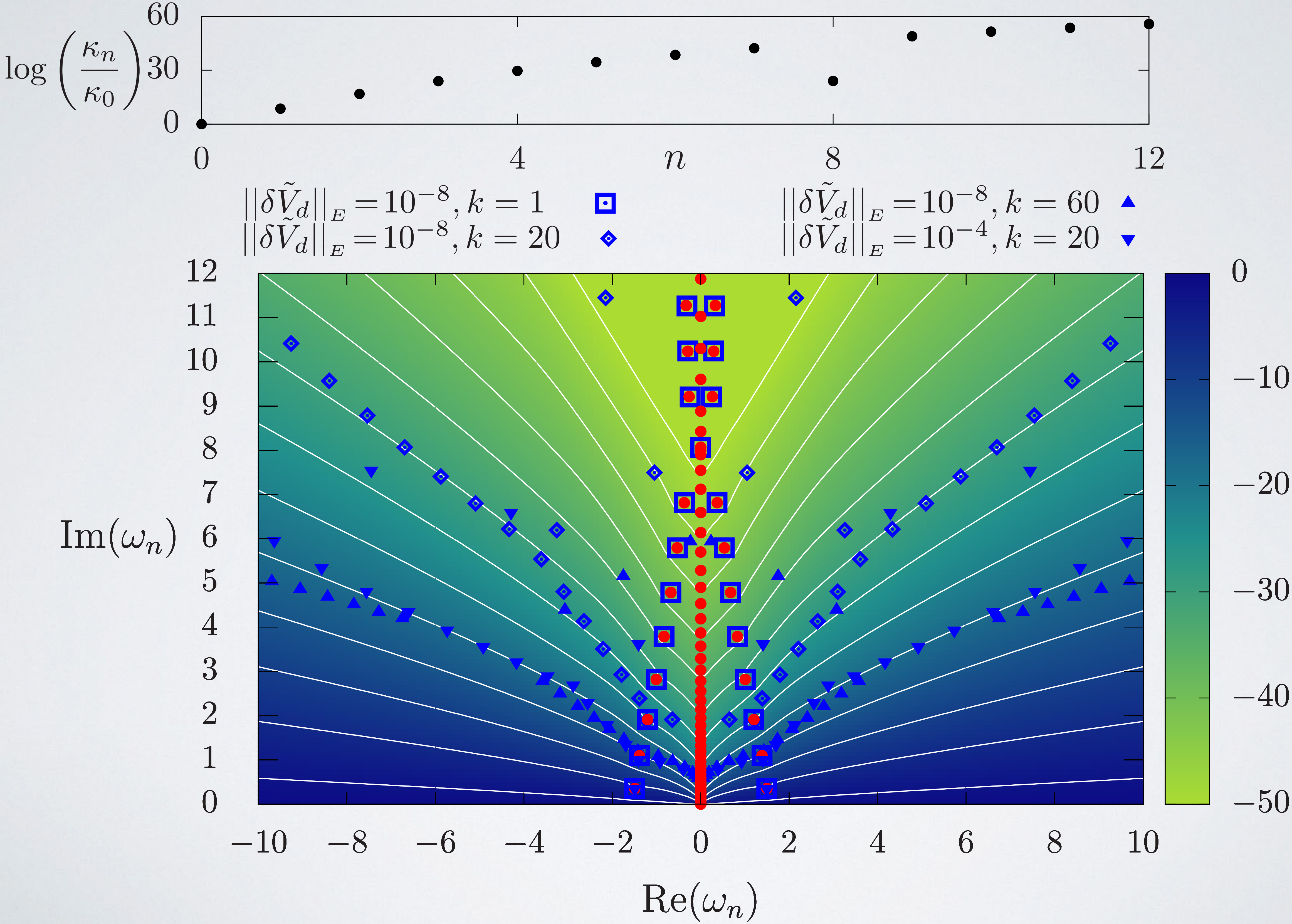
Partial Conclusions:

- Condition number and pseudospectra indicate potential instabilities under perturbation of the operator.
- Physical scenario: *ad hoc* modification of the potential



SPECTRAL INSTABILITY

Perturbed potential: $q_\ell \to q_\ell + \delta V_\mathrm{d}$ $\quad \delta V_\mathrm{d} \propto \cos(2\pi k\,\sigma)$



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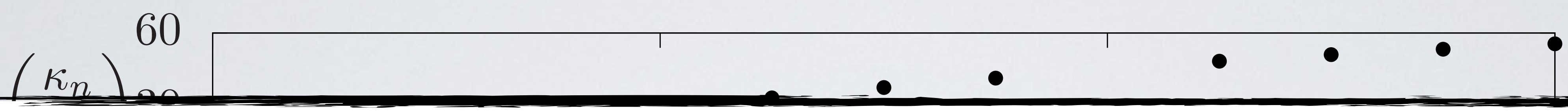
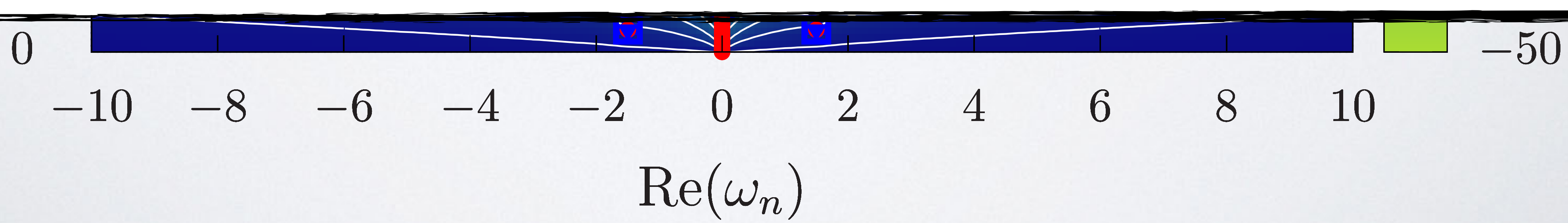


Partial Conclusions:

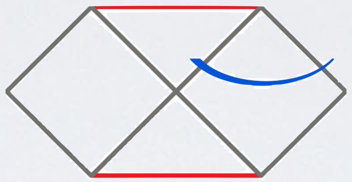
- Fundamental Mode is stable (potential perturbation respects asymptotic decays: ultra-violet effect)
- Overtones are stable under low wave number perturbation
- Overtones are unstable under high wave number perturbations
- Fundamental Mode is unstable (potential perturbation disturbs asymptotic decays: infra-red effect)

"the elephant and the flea effect- M. Cheung, K. Destounis, RPM, E. Berti, V. Cardoso. PRL 128 11 (2022)"



CONCLUSION

- **Mindset change:** Think spacetime and use the coordinates most adapted to the problem.



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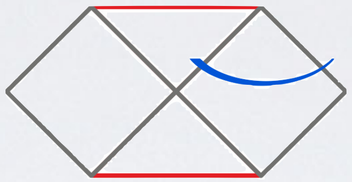

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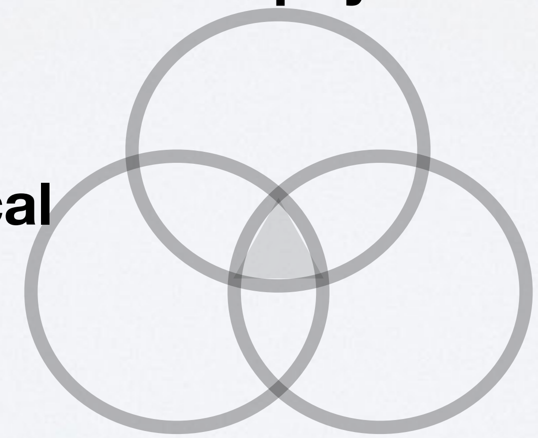
- **XXI Century BH Pert. Theory:** Beyond the linear regime (see A. Pound talk)



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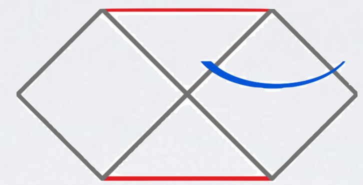

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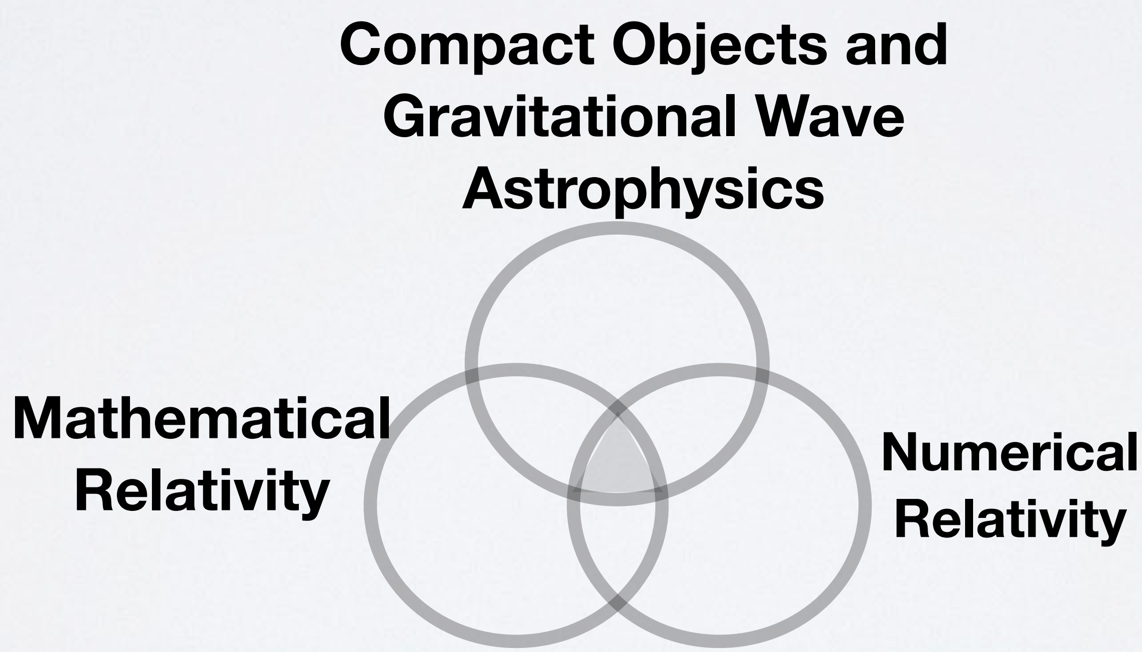

- **Tools from non-hermitian physics:** Introduce pseudosepctra and QNM condition number to GR, addressing a potential crucial issue in GW astronomy: QNM instability