COMPARING HERMITIAN AND NON-HERMITIAN QUANTUM ELECTRODYNAMICS

Jake Southall, **Daniel Hodgson**, Rob Purdy and Almut Beige

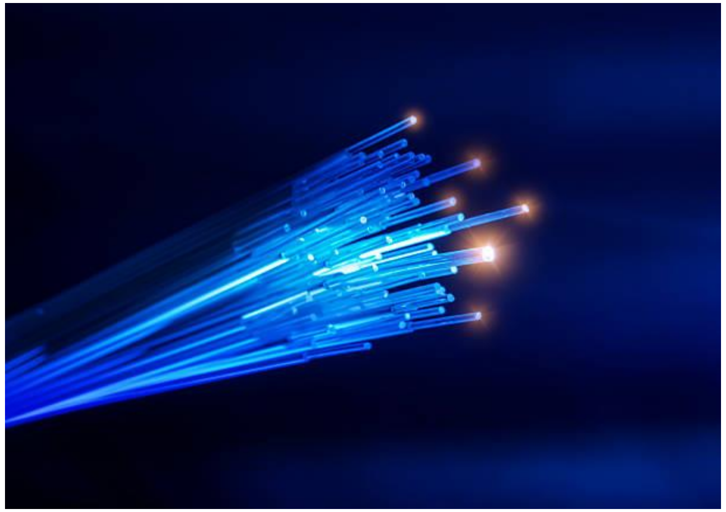
University of Leeds, UK

King’s College London, March 2025

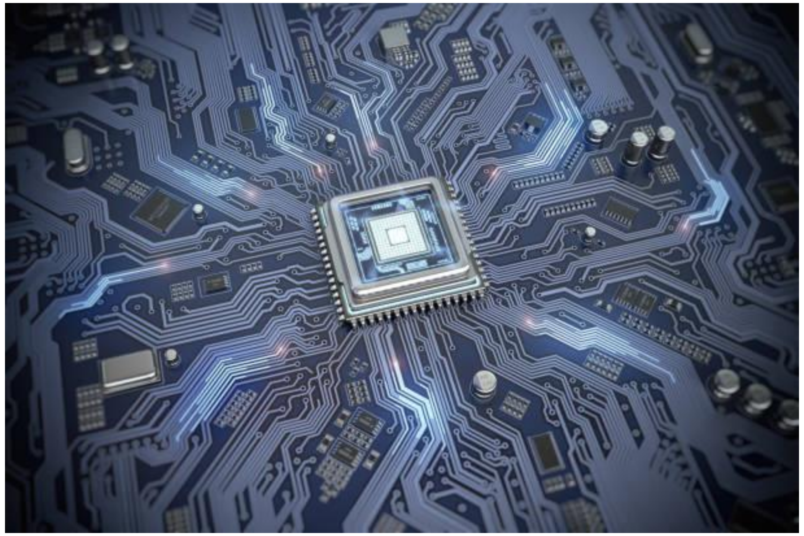
Quantum models of light

Light is a versatile resource with many applications.

Quantum Sensing



Communications



Quantum algorithms

Having a complete theory is crucial for describing and understanding complex dynamics.

Image: https://www.idquantique.com/quantum-sensing/applications/materials-science/

Electromagnetism in 1D

Maxwell's equations for 1D fields in the absence of charges:

$$\frac{\partial}{\partial x}\mathrm{E}_{\mathrm{V(H)}}(x,t) = \pm\frac{\partial}{\partial t}\mathrm{B}_{\mathrm{H(V)}}(x,t)$$

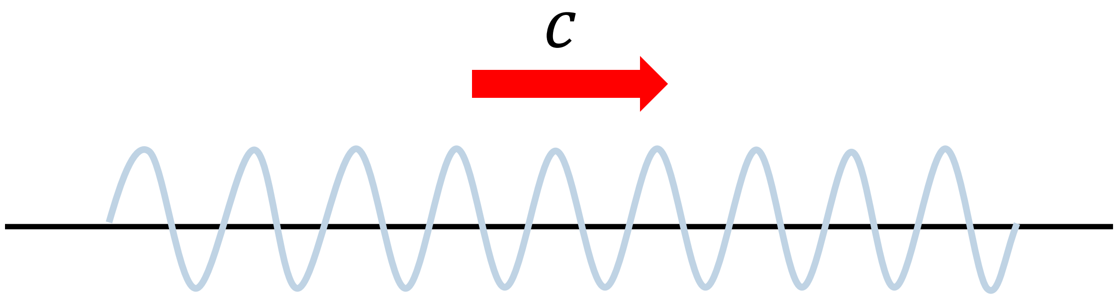
$$c^2\frac{\partial}{\partial x}\mathrm{B}_{\mathrm{H(V)}}(x,t) = \pm\frac{\partial}{\partial t}\mathrm{E}_{\mathrm{V(H)}}(x,t)$$

c - Speed of light

H (V) – Horizontally (Vertically) polarised

James Clerk-Maxwell
1831-1879

Monochromatic states



$$|1_{s\lambda}(k,t)\rangle = a^{\dagger}_{s\lambda}(k,t)|0\rangle$$

$a^{\dagger}_{s\lambda}(k,t)$ – creation operator

$|0\rangle$ - vacuum state

$k \geq 0$

Orthogonality:

$$\langle 1_{s\lambda}(k,t)|1_{s'\lambda'}(k',t)\rangle = \delta_{s,s'}\,\delta_{\lambda,\lambda'}\,\delta(\omega-\omega')$$

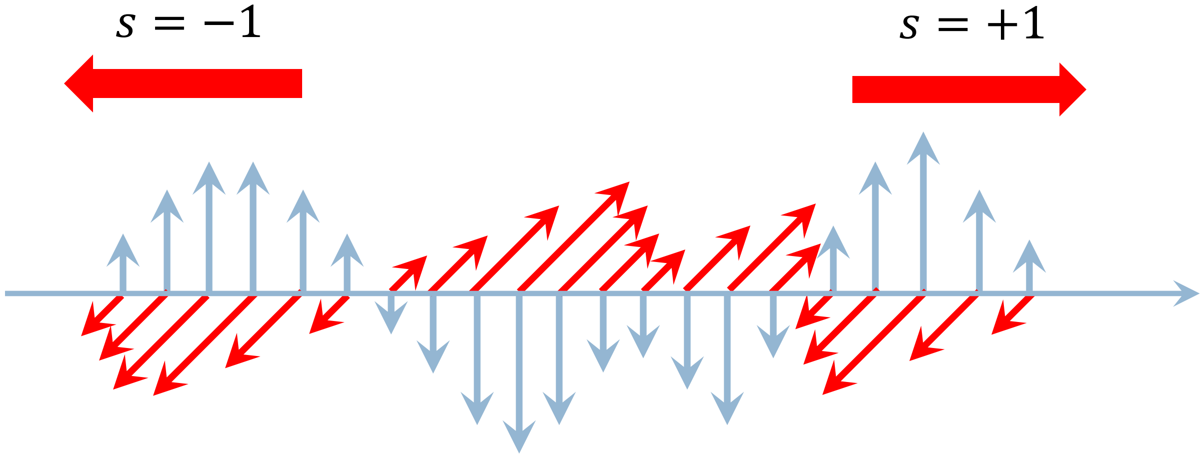
Monochromatic excitations are distinct and evolve independently.

Electromagnetic waves

Maxwell's equations describe the propagation of a wave

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathrm{E}_\lambda(x,t) = 0 \quad \Longrightarrow \quad \left[\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right]\left[\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right]\mathrm{E}_\lambda(x,t) = 0$$

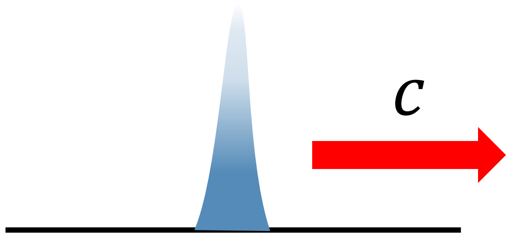
Jean le Rond D'Alembert
1717-1783



Solution: $$\mathrm{E}(x,t) = \sum_{s=\pm 1} \mathrm{E}_s(x - sct)$$

Bosons localised in position (blip)

A blip is a localised excitation of the EM field.



$$|1_{s\lambda}(x,t)\rangle = a^{\dagger}_{s\lambda}(x,t)|0\rangle$$

$$a^{\dagger}_{s\lambda}(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{iskx} a^{\dagger}_{s\lambda}(k,t)$$

Increases the number of degrees of freedom

Orthogonality

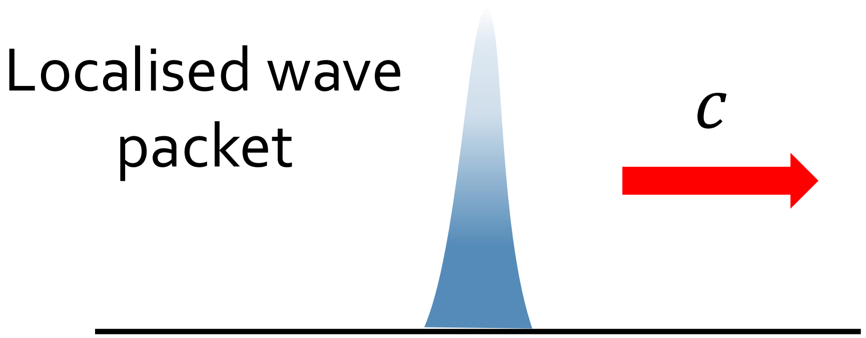
$$\langle 1_{s\lambda}(x,t)|1_{s'\lambda'}(x',t)\rangle = \delta_{s,s'}\,\delta_{\lambda,\lambda'}\,\delta(x-x')$$

Blips are **localised** and evolve independently.

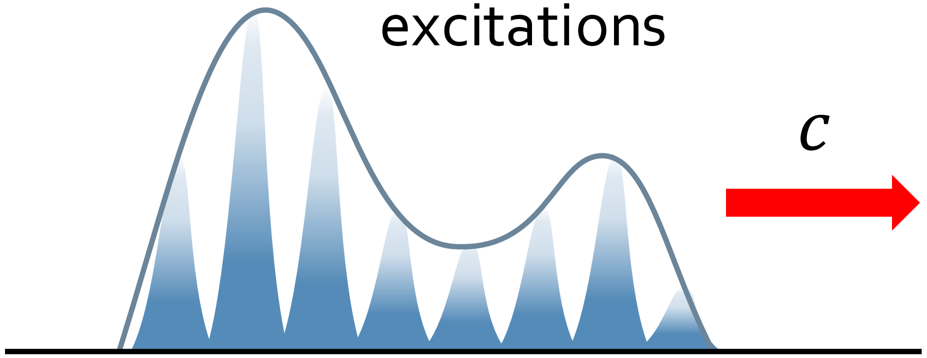
Position and frequency

Monochromatic waves are an ideal solution and have no defined position.

Alternatively, light can be composed of excitations that are localised.



Single-photon wave packet:
Normalised superposition of excitations



Aim: to construct a theoretical model of localised excitations.

Electric and magnetic field observables

The EM field observables $\mathrm{E}(x, t)$ and $\mathrm{B}(x, t)$ determine the strength of the electric and magnetic fields respectively at a position x and a time t.

$$\mathrm{E}(x,t) = \sum_{s=\pm 1} \int_{-\infty}^{\infty} dx' \; c\,\Re(x-x')[a_{sH}(x',t)\,\hat{\mathbf{y}} + a_{sV}(x',t)\,\hat{\mathbf{z}}] + \mathrm{H.c}$$

$$\mathrm{B}(x,t) = \sum_{s=\pm 1} \int_{-\infty}^{\infty} dx' \; s\,\Re(x-x')[-a_{sV}(x',t)\,\hat{\mathbf{y}} + a_{sH}(x',t)\,\hat{\mathbf{z}}] + \mathrm{H.c}$$

$\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ - unit polarisation vectors

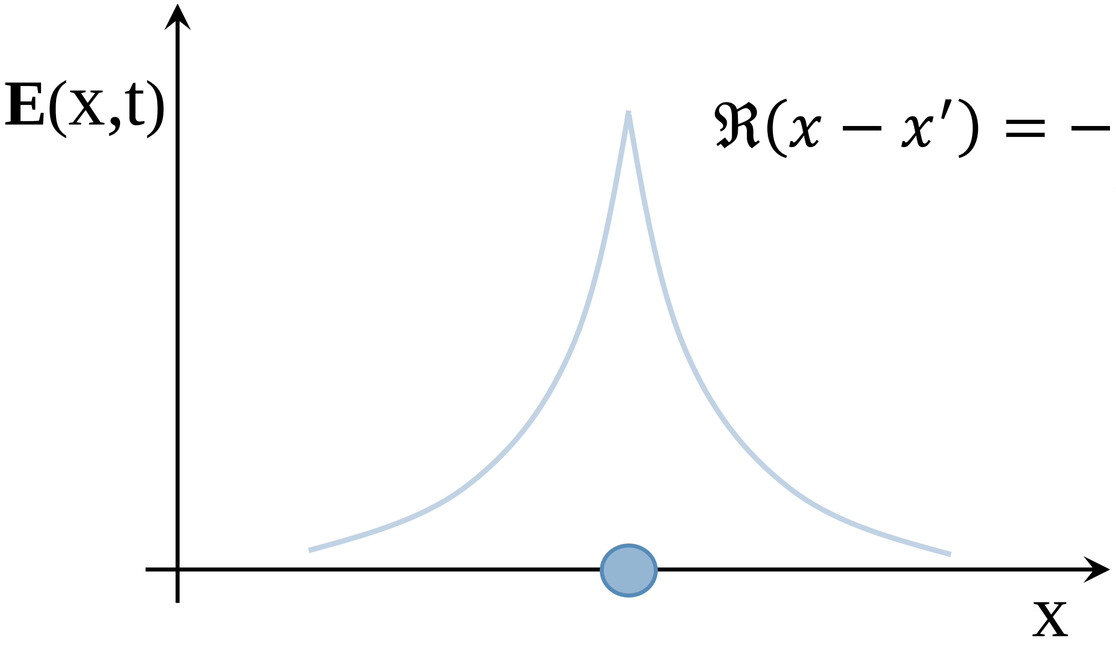
$\Re(x - x')$ – regularisation function

$\Re(x - x')$ determines how the field responds to blips at different positions.

Non-locality

Fourier transforms: $\tilde{E}(k,t) \sim \sqrt{|k|}\, a_{s\lambda}(k,t)$

The regularisation function introduces non-locality:

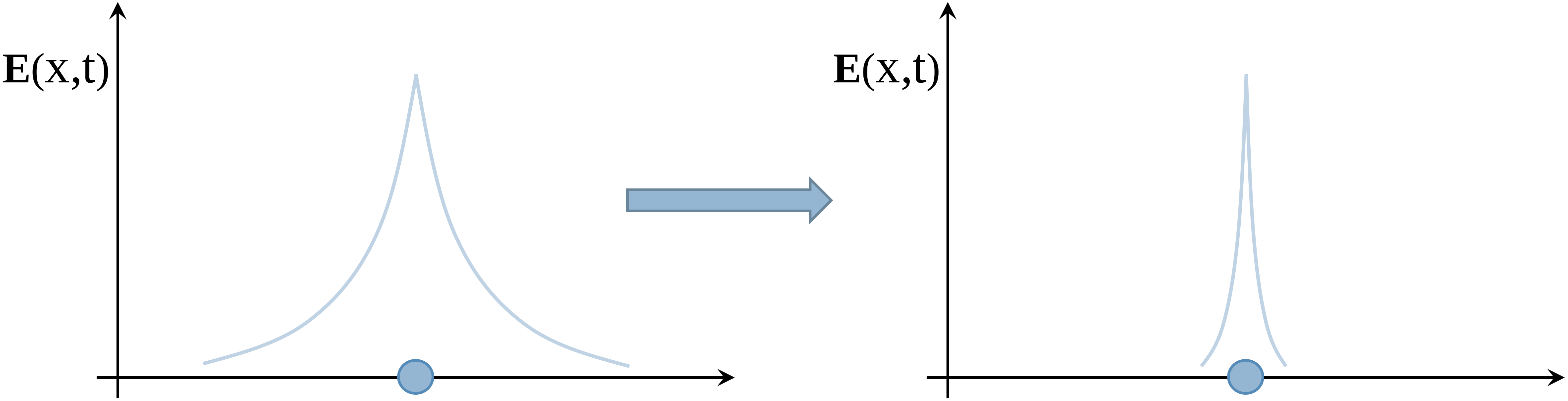


$$\Re(x - x') = -\sqrt{\frac{\hbar}{4\pi\varepsilon c A}} \cdot \frac{1}{|x - x'|^{3/2}}$$

A – area of field
ε – permittivity

Blips carry a non-local field!

Alternatively: Field localisation



Standard Inner product

Biorthogonal Inner product

Field localisation

By looking at the field observables, the states that are locally related to the field observables are

$$\left|1_{s\lambda}^{\text{field}}(x,t)\right\rangle = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \sqrt{|k|}\, e^{iskx}\, a_{s\lambda}^{\dagger}(k,t)|0\rangle$$

Not orthogonal!

By introducing a generalised **biorthogonal inner product** we can enforce that

$$\left\langle 1_{s\lambda}^{\text{field}}(x,t)^{\text{bio}} \middle| 1_{s'\lambda'}^{\text{field}}(x',t)\right\rangle = \delta_{ss'}\,\delta_{\lambda\lambda'}\,\delta(x-x')$$

and $$\left\langle 1_{s\lambda}(k,t)^{\text{bio}} \middle| 1_{s\lambda}(k',t)\right\rangle = \delta_{ss'}\,\delta_{\lambda\lambda'}\,\delta(k-k')$$

Biorthogonal quantum physics

Hermitian Physics:

An orthogonal basis $\{|\alpha_n\rangle\}$ can be found under the standard inner product such that

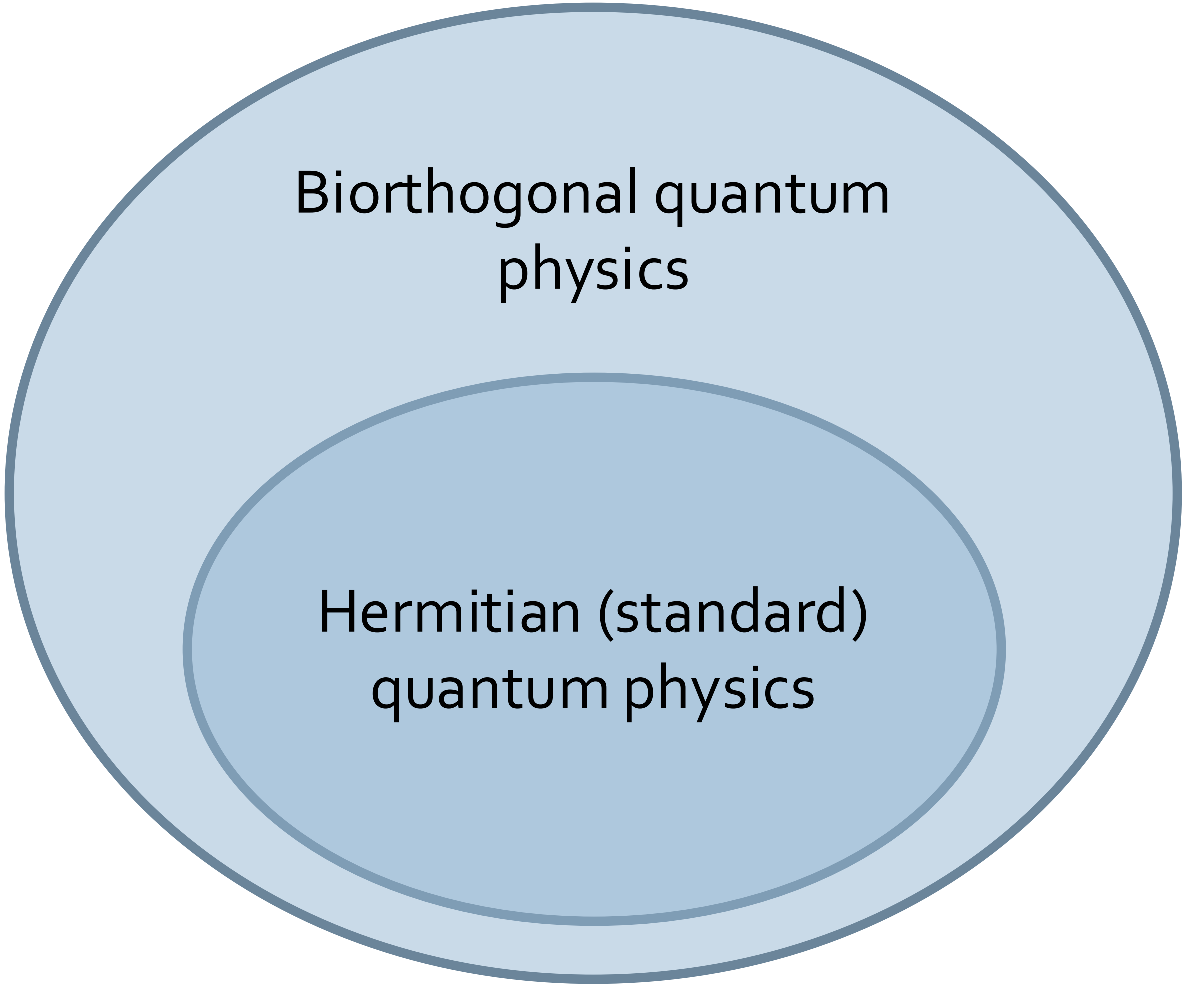
$$\langle\alpha_n|\alpha_m\rangle = \delta_{nm}$$

Biorthogonal physics:

For a **chosen** set of states $\{|\alpha_n\rangle\}$, a biorthogonal system $\{|\beta_n\rangle, |\alpha_n\rangle\}$ can be found such that

$$\langle\beta_n|\alpha_m\rangle = \delta_{nm}$$

Generalised inner products



Biorthogonal systems motivate a generalised inner product

$$\langle \alpha_n^{\mathrm{bio}} | \alpha_m \rangle = \langle \beta_n | \alpha_m \rangle$$

By redefining the inner product we can choose an alternative orthogonal basis.

Biorthogonal quantum electrodynamics

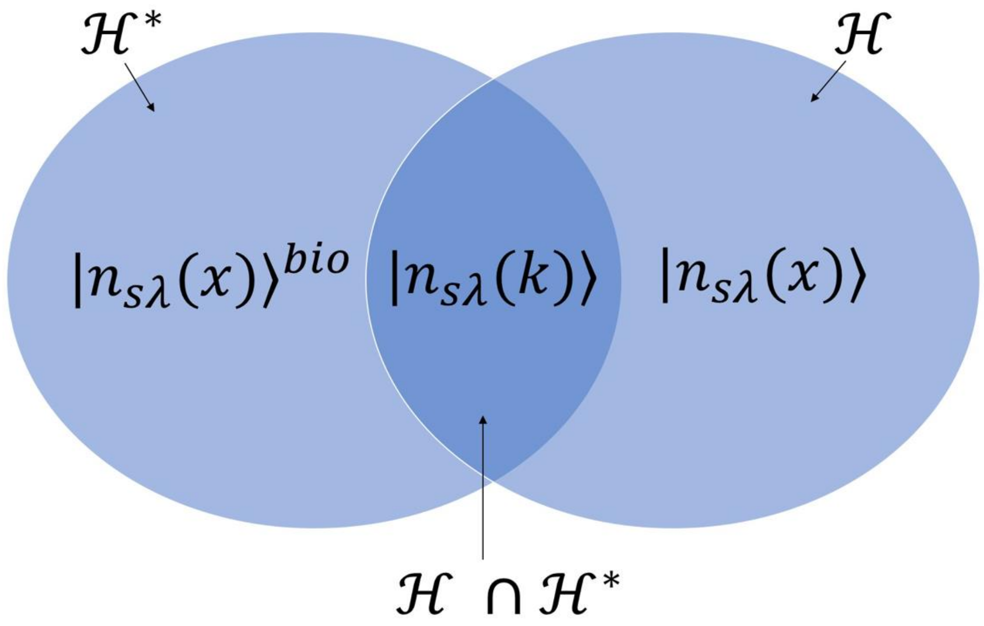
Local field excitations:

$$\left|1_{s\lambda}^{\text{field}}(x,t)\right\rangle \qquad \left|1_{s\lambda}^{\text{field}}(x,t)^{\text{bio}}\right\rangle$$

Monochromatic field excitations:

$$|1_{s\lambda}(k,t)\rangle$$

Monochromatic excitations are naturally biorthogonal



Photon dynamics

Time-evolved state:

$$|\psi(t)\rangle = U(t,0)|\psi(0)\rangle$$

Time-evolution operator:

$$U(t,0) = \exp\left[-iH_{dyn}t/\hbar\right]$$

In a closed quantum system, the inner product must be preserved.

$$\left\langle \psi(t)^{\mathrm{bio}} \middle| \phi(t) \right\rangle = \left\langle \psi(0)^{\mathrm{bio}} \middle| \phi(0) \right\rangle$$

$$H_{dyn} = H_{dyn}^{\dagger(\mathrm{bio})}$$

The bio-Hermitian conjugate is in general **different** to the standard Hermitian conjugate.

Locally-interacting Hamiltonians

The dynamical Hamiltonian for a free photon is both Hermitian and bio-Hermitian

$$H_{dyn} = H_{dyn}^{\mathrm{bio}}$$

When there is a local field interaction, however, the Hamiltonian is bio-Hermitian only.

$$H_{int} = \sum_{s=\pm 1} a_{s\lambda}^{\dagger}(0)\, a_{-s\lambda}^{\mathrm{bio}}(0)$$

$$H_{int} \neq H_{int}^{\mathrm{bio}}$$

Pseudo-Hermitian operators

When we introduce a new inner product, the definition of Hermiticity changes

$$\langle\psi|\phi\rangle^{bio} = \langle\psi|\eta|\phi\rangle \qquad \eta - \text{Hermitian operator}$$

An operator O is Hermitian when

$$\langle O\psi|\phi\rangle^{\text{bio}} = \langle\psi|O\phi\rangle^{\text{bio}} \quad \Longrightarrow \quad O = \eta O^{\dagger}\eta^{-1}$$

O still has real eigenvalues even though in general $O \neq O^{\dagger}$

Conclusions and Remarks

- We can change the scalar product of quantum physics to make non-orthogonal state appear orthogonal.

$$\langle\psi|\phi\rangle^{bio} = \langle\psi|\eta|\phi\rangle$$

- Then we also need to change Hamiltonian and observables to keep the same physical meaning and dynamics.

$$\langle O\psi|\phi\rangle^{\mathrm{bio}} = \langle\psi'|O|\psi'\rangle$$

At the end, nothing has changed!

Reference:
Comparing Hermitian and non-Hermitian Quantum Electrodynamics,
J. Southall, D. Hodgson, R. Purdy, and A. Beige, Symmetry 14, 1816 (2022).