# Quasi-normal mode expansion of black hole perturbation: a hyperboloidal Keldysh approach

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**arXiv:2412.02793 with JL. Jaramillo[1]**

and ongoing work with P. Bizoń*, V. Boyanov, JL. Jaramillo*, D. Pook-Kolb*

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Common trends in non-Hermitian Physics: Black Holes and Quantum Optics, March 2025

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# Introduction

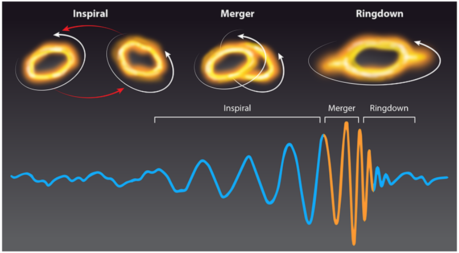



Figure: Illustration of a pair of coalescing black holes (credit : (Top) Kip Thorne; (Bottom) B. P. Abbott et al; adapted by APS/Carin Cain)

## Ringdown

Sum of damped oscillators

$$\Psi_{\ell,m}(t) \sim \sum_n A_{\ell,m,n} e^{i\omega_{\ell,m,n} t}$$

# Introduction : the conservative case

## The conservative case

Example of system : guitar string struck
Consider the linear equation

$$\begin{cases} \partial_t u = iHu \\ u(t=0,x) = u_0(x) \end{cases}$$

where $H$ is self-adjoint and the eigenfunctions $\hat{v}_n$ form an orthonormal basis of the Hilbert space. The solution can be written as a convergent sum (convergent series) over the harmonics

$$u(x,t) = \sum_{n=0}^{\infty} a_n \hat{v}_n(x) e^{i\omega_n t}$$

where

$$a_n = \langle \hat{v}_n, u_0 \rangle_{\mathsf{G}}, \qquad H\hat{v}_n = \omega_n \hat{v}_n$$

# The non-normal case

## (In)Completeness

In the **self-adjoint** (normal) case, any oscillation is a superposition of normal modes. In the **non-self adjoint** (non-normal) case, we don't have completeness for **generic** potentials[i].

## Spectral decomposition, the need for a systematic approach

Spectral decomposition and excitation coefficients in the Schwarzschild case[ii].

## Keldysh expansion through the adjoint of $L$

Keldysh expansion from $L$ and $L^\dagger$ introduced in previous work[iii].

$$\begin{cases} Lv_n = \omega_n v_n \\ L^\dagger w_n = \overline{\omega_n} w_n \end{cases} \qquad u(x,\tau) \sim \sum_n a_n^{\mathsf{G}} \hat{v}_n(x) e^{i\omega_n \tau} \qquad \text{where } a_n^{\mathsf{G}} = \frac{\langle w_n, u_0 \rangle_{\mathsf{G}}}{\langle w_n, v_n \rangle_{\mathsf{G}}}$$

[i] Warnick, (In)completeness of Quasinormal Modes, Acta Physica Polonica B

[ii] Ansorg, Macedo, Spectral decomposition of black-hole perturbations on hyperboloidal slices, Phys. Rev. D 93, 124016

[iii] Gasperin, Jaramillo, Energy scales and black hole pseudospectra: the structural role of the scalar product, Class. Quantum Grav. 39 115010

# Introduction : quasinormal modes definitions and BH perturbation theory

## Quasi-normal modes (QNMs)

Heuristics: Resonant response under linear perturbation characterized by complex frequencies. QNMs probe the background spacetime geometry
Lax-Phillips theory: QNMs as poles of the resolvent

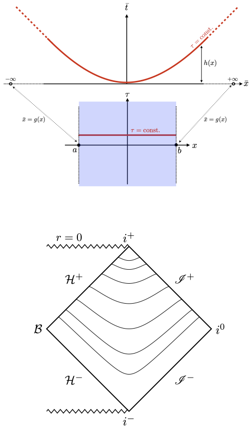
## Perturbation theory on Schwarzschild black hole

Scalar, electromagnetic and gravitational perturbations reduce to the following wave equation in the tortoise coordinates $(t, r_*)$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {r_*}^2} + V_\ell(r_*)\right)\phi_{\ell m} = 0 + \text{outgoing boundary conditions}$$

where $V_\ell$ depends on the type of perturbation (spin $s = 0, 1$ or $2$).

# Compactified hyperboloidal approach




Illustrations of hyperboloidal slicings[iv]

## Compactified hyperboloidal approach

$$\begin{cases} t = \tau - h(x) \\ r_* = g(x) \end{cases} \qquad \partial_\tau u(x,\tau) = iLu(x,\tau)$$

## Toy model[v]

$$V(r_*) = \operatorname{sech}^2(r_*) \qquad \begin{cases} \tau = t - \ln(\cosh r_*) \\ x = \tanh^{-1}(r_*) \end{cases}$$

Analytic solutions: the eigenvectors $v_n(x)$ of $L$ are Gegenbauer polynomials and $\omega_n = \pm\frac{\sqrt{3}}{2} + i\left(n + \frac{1}{2}\right)$

[iv] PhysRevX.11.031003, Jaramillo, Macedo, Al Sheikh and hyperboloid.al

[v] Al Sheikh,Jaramillo,Macedo;2004.06434,Al Sheikh PhD,Bizoń,Chmaj,Mach;2002.01770

# Compactified hyperboloidal approach

## Compactified hyperboloidal slicing

$$\begin{cases} t = \tau - h(x) \\ r_* = g(x) \end{cases} \qquad \begin{aligned} g\colon [a,b] &\to [-\infty, +\infty] \\ x &\mapsto g(x) = r_* \end{aligned}$$

## First order reduction

We define the field $\psi := \partial_\tau \phi$, the linear problem becomes

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right) \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \qquad \text{Pöschl-Teller} : \begin{cases} L_1 = \partial_x((1-x^2)\partial_x) - V_0 \\ L_2 = -(2x\partial_x + 1) \end{cases}$$

## Outgoing boundary conditions

- Geometric interpretation : outgoing null cones
- Analytic interpretation : singular Sturm-Liouville operator, the boundary conditions are built-in as regularity conditions

# Compactified hyperboloidal approach

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}_{iL} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

All the space derivatives are contained within the matrix.

$$L_1 = \frac{1}{w(x)} \left(\partial_x (p(x) \partial_x) - q(x)\right)$$

$$L_2 = \frac{1}{w(x)} \left(2\gamma(x)\partial_x + \partial_x \gamma(x)\right)$$

where

$$w(x) = \frac{g'(x)^2 - h'(x)^2}{|g'(x)|}$$

$$p(x) = \frac{1}{|g'(x)|}$$

$$q(x) = |g'(x)| V_\ell(x)$$

$$\gamma(x) = \frac{h'(x)}{|g'(x)|}$$

## 4 cases of study

i) Pöschl-Teller
ii) Schwarzschild
iii) Schwarzschild-de Sitter
iv) Schwarzschild-Anti de Sitter

The Schwar.-AdS case corresponds to a **generalized eigenvalue problem**.

# Scalar product and non-selfadjointness

The energy scalar product is related to the energy-momentum tensor of a complex scalar field on a Minkowski spacetime with a potential $V_\ell$.

$$\left\langle \begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix} \right\rangle_E = \frac{1}{2}\int_a^b w(x)\overline{\psi}_1\psi_2 + p(x)\partial_x\overline{\phi}_1\partial_x\phi_2 + q_\ell(x)\overline{\phi}_1\phi_2 dx$$

We use this to justify that $L_2$ is a dissipative term and is responsible for non-self adjointness

$$L^\dagger = L + \frac{1}{i}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 2\frac{\gamma(x)}{w(x)}\left(\delta(x-a)-\delta(x-b)\right)\end{array}\right)$$

## Instability of the QNMs

The eigenvalues can be greatly perturbed upon a small perturbation of the potential

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# Resolvent

$$\partial_\tau u(x,\tau) = iLu(x,\tau)$$

A Laplace transform yields,

$$(L-\omega)u(x,\omega) = iu_0(x)$$

Acting with the resolvent $R_L(\omega) = (L-\omega I)^{-1}$ on both sides we get

$$u(x,\omega) = i(L-\omega I)^{-1}u_0(x)$$

# Keldysh's expansion of the resolvent

Consider the application

$$\begin{aligned} F\colon \Omega &\to \mathcal{L}(\mathcal{H},\mathcal{K}) \\ \omega &\mapsto F(\omega) \end{aligned}$$

Assume $F(\omega)$ is a Fredholm operator. The transpose application of $F$ is

$$F(\omega)^t\colon \mathcal{K}^* \to \mathcal{H}^*$$

The spectral problems are rewritten

$$F(\omega_n)v_n = 0, \qquad F(\omega_n)^t\alpha_n = 0, \qquad v_n \in \mathcal{H}, \alpha_n \in \mathcal{K}^*$$

Keldysh's theorem gives an expansion of the resolvent application[vi].

$$\boxed{F^{-1}(\omega) = \sum_{\omega_n\in\Omega_0} \frac{\langle \widetilde{\alpha}_n, .\rangle}{\omega-\omega_n} v_n + H(\omega) \quad \text{with } \left\langle \widetilde{\alpha}_n, \frac{dF}{d\omega}(\omega_n)(v_n) \right\rangle = 1.} \tag{1}$$

[vi] Beyn, Latushkin, Rottmann-Matthes; 1210.3952

# Keldysh's resonant expansion for non-generalized eigenvalue problems

We use the recipe with $F(\omega) = L - \omega I$, the spectral problems are :

$$(L - \omega_n I)v_n = 0, \qquad (L^t - \omega_n I)\alpha_n = 0, \qquad v_n \in \mathcal{H}, \alpha_n \in \mathcal{H}^*$$

The resolvent of $L$ is constructed in a bounded domain $\Omega$ can be written

$$R_L(\omega) = (L - \omega I)^{-1} = \sum_{\omega_n \in \Omega_0} \frac{\langle \widetilde{\alpha}_n, . \rangle}{\omega - \omega_n} v_n + H(\omega)$$

On the other hand, the Laplace transform of the differential equation yields

$$(L - \omega)u(x, \omega) = iu_0(x)$$

The asymptotic resonant expansion is then found by multiplication by the resolvent and inverse Laplace transform

$$\boxed{u(\tau, x) \sim \sum_n \langle \alpha_n, u_0 \rangle v_n(x) e^{i\omega_n \tau}, \qquad \text{with } \langle \alpha_n, v_n \rangle = 1}$$

# Asymptotic resonant expansion

## Bound of the error of the Keldysh expansion

Given a bounded domain $\Omega$ in $\mathbb{C}$, we have

$$u(\tau, x) = \sum_{n=0}^{N_{\mathsf{QNM}}} \langle \alpha_n, u_0 \rangle v_n(x) e^{i\omega_n \tau} + E_{N_{\mathsf{QNM}}}(\tau; u_0)(x) \quad \text{with } \langle \alpha_n, v_n \rangle = 1$$

$$\boxed{\|E_{N_{\mathsf{QNM}}}(\tau; u_0)\| \leq \|u_0\| C(N_{\mathsf{QNM}}, L) e^{-\operatorname{Im}\left\{\omega_{N_{\mathsf{QNM}}}\right\}\tau}}$$

## $\mathcal{A}_n(x)$ or $a_n^{\mathsf{G}} \hat{v}_n(x)$ ?

$$\mathcal{A}_n(x) := \langle \alpha_n, u_0 \rangle v_n(x) = \underbrace{\langle w_n, u_0 \rangle_{\mathsf{G}}}_{a_n^{\mathsf{G}}} \hat{v}_n(x), \quad \text{with } \langle \alpha_n, v_n \rangle = \langle w_n, \hat{v}_n \rangle_{\mathsf{G}} = 1$$

$a_n^{\mathsf{G}}$ depends on the norm but $\mathcal{A}_n(x)$ doesn't.

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# Numerical methods, Chebyshev's interpolation

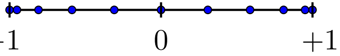



Figure: Chebyshev-Lobatto grid, $x_j = \cos\left(\frac{\pi j}{N}\right)$

The discretized counterpart of $\phi$ is a vector with $N+1$ entries:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \\ \phi_{N+1} \end{bmatrix}, \qquad \phi(x_j) = \phi_j$$

<u>Numerical instability</u>: we work with arbitrary precision numerics, typically $10^{-800}$

$L : (2N+2) \times (2N+2)$ entries (matrix)

$u : (2N+2)$ entries (column vector)

# Time evolutions

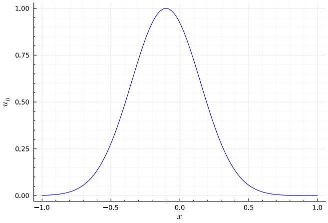



Figure: Initial data

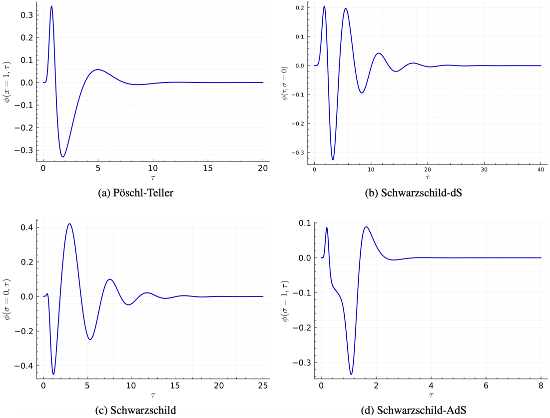
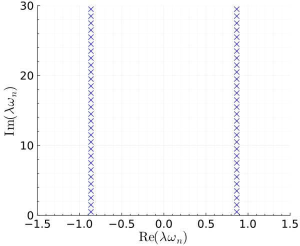
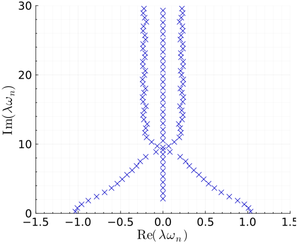



Figure: Waveforms at future null infinity (event horizon for the AdS case)

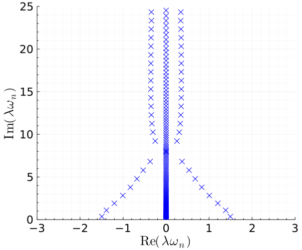
# Spectra



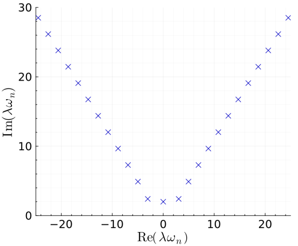

(a) Pöschl-Teller




(b) Schwarzschild-dS




(c) Schwarzschild




(d) Schwarzschild-AdS

# Spectra

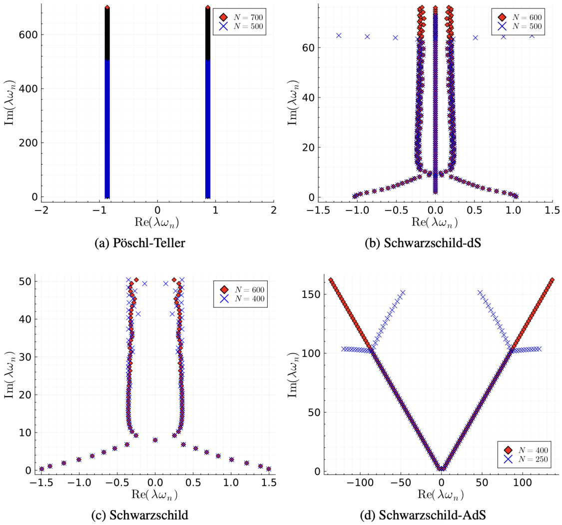
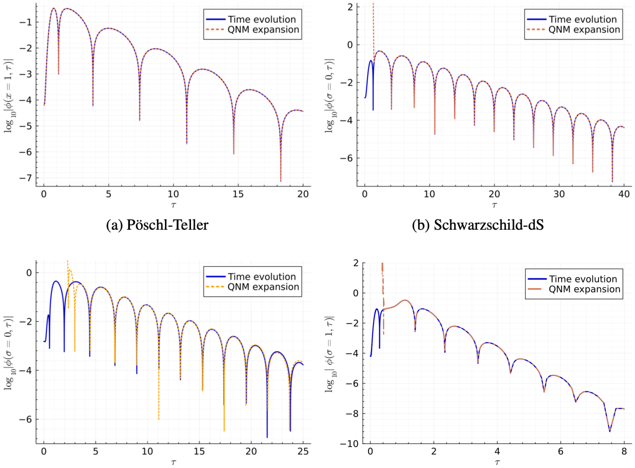



Figure: Spectra of the cases of study for different gridsizes $N$.

# Comparing the time and the spectral domain analysis

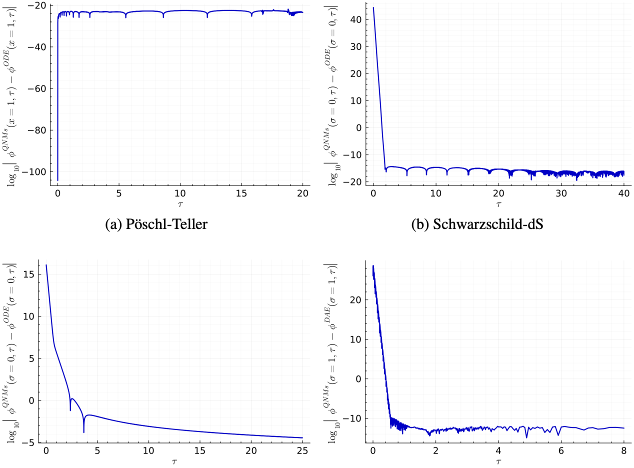



(a) Pöschl-Teller (b) Schwarzschild-dS

(c) Schwarzschild (d) Schwarzschild-AdS

Figure: We compare the ODE solution and the Keldysh QNM expansion at future null infinity (event horizon for the AdS case).

# Comparing the time and the spectral domain analysis




(a) Pöschl-Teller

(b) Schwarzschild-dS

(c) Schwarzschild

(d) Schwarzschild-AdS

Figure: Difference between the ODE solution and the Keldysh QNM expansion at future null infinity.

# Coefficients of the timeseries at future null infinity

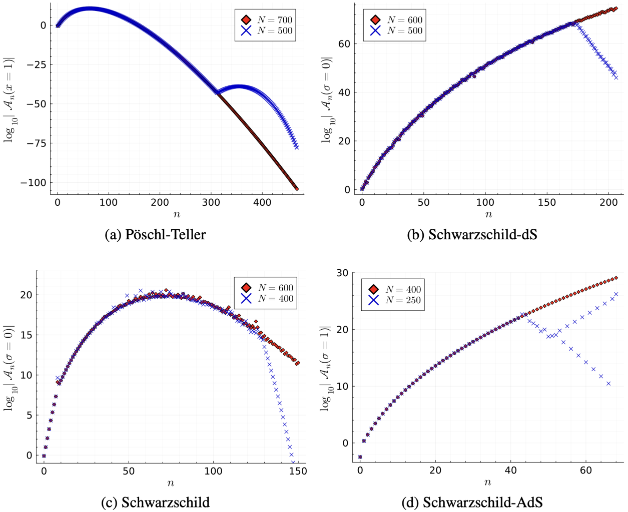
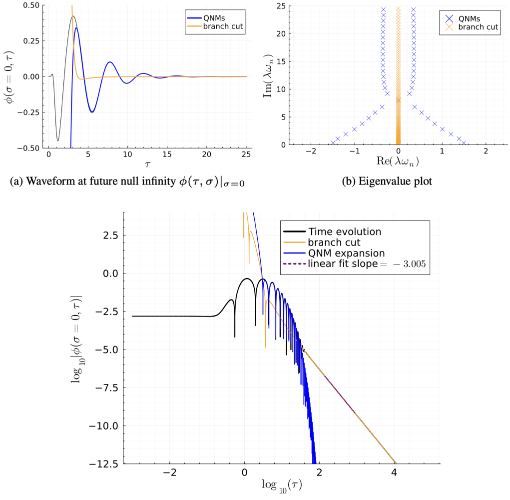



Figure: Log plot of the modulus of the coefficients $\mathcal{A}_n^{\infty}$.

# Schwarzschild case : separating tails and QNMs



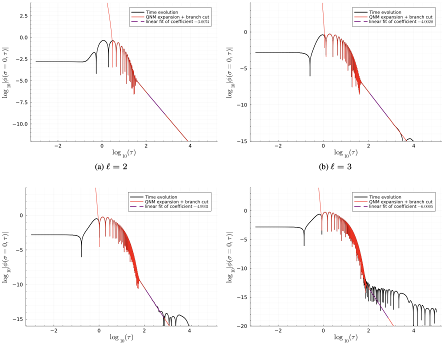

(a) Waveform at future null infinity $\phi(\tau, \sigma)|_{\sigma=0}$

(b) Eigenvalue plot

(c) Waveform $\phi(\tau, \sigma)|_{\sigma=0}$ on a log-log plot

Figure: Separating branch cut and QNMs in the Schwarzschild case.

# Polynomial tails and branch cut




(a) $\ell = 2$ (b) $\ell = 3$ (c) $\ell = 4$ (d) $\ell = 5$

Figure: Polynomial tails in the Schwarzschild case.

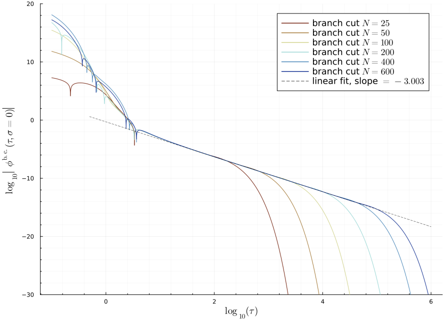



Figure: The bigger $N$, the longer the tail. ($\ell = 2$ in this figure).

# Dynamics from the (exponentiated) evolution operator

## Finite rank case (matrix case)

We assume the matrix $\boldsymbol{L}$ can be diagonalized: $\boldsymbol{L} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ where

$$\begin{aligned}
\boldsymbol{D} &= \mathrm{diag}(\omega_1, \omega_2, ..., \omega_{2N+1}, \omega_{2N+2}) \\
\boldsymbol{P} &= \left( \ \boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid ... \mid \boldsymbol{v}_{2N+1} \mid \boldsymbol{v}_{2N+2} \ \right) \\
\left(\boldsymbol{P}^{-1}\right)^t &= \left( \ \boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid ... \mid \boldsymbol{\alpha}_{2N+1} \mid \boldsymbol{\alpha}_{2N+2} \ \right)
\end{aligned}$$

- $\boldsymbol{v}_n$ are eigenvectors of $\boldsymbol{L}$
- $\boldsymbol{\alpha}_n$ are eigenvectors of $\boldsymbol{L}^t$ such that $\langle \boldsymbol{\alpha}_n, \boldsymbol{v}_n \rangle = 1$

The "formal" solution of $\partial_\tau \boldsymbol{u} = i\boldsymbol{L}\boldsymbol{u}$ is $e^{i\boldsymbol{L}\tau}\boldsymbol{u}_0$, it corresponds to the sum over **all** the eigenvalues,

$$\boldsymbol{P}e^{i\boldsymbol{D}\tau}\boldsymbol{P}^{-1}\boldsymbol{u_0} = \sum_{n=1}^{2N+2} \frac{\langle \boldsymbol{\alpha}_n, \boldsymbol{u_0} \rangle}{\langle \boldsymbol{\alpha}_n, \boldsymbol{v_n} \rangle} e^{i\omega_n \tau} \boldsymbol{v_n}$$

# Role of overtones: Pöschl-Teller

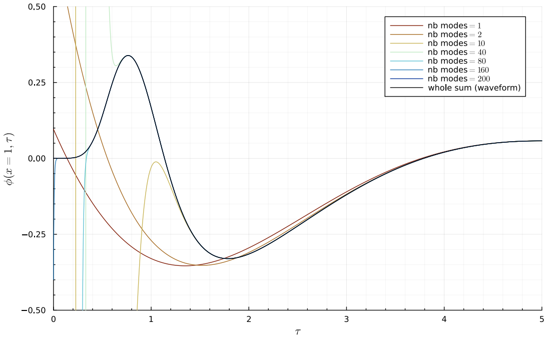



Figure: We show we recover the early times of the waveform by adding enough overtones. The panel on the right is a zoom.

<u>Convergence of the series ?</u> We can describe the waveform using $310$ modes with a maximum error $\approx 10^{-40}$, this begs the question whether the series is convergent or not. What meaning do we give to the word "convergent" here ?

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# Pseudospectrum

Given a perturbation $\delta L$ of $L$ of norm $\varepsilon$, what is the set of complex numbers $\lambda$ which are actual eigenvalues of some perturbed operator $L + \delta L$ ?

## Perturbative approach

$$\sigma^{\varepsilon}(L) = \{\lambda \in \mathbb{C}, \exists \delta L \in M_n(\mathbb{C}), \|\delta L\| < \varepsilon : \lambda \in \sigma(L + \delta L)\}$$

## Resolvent norm approach

$$\sigma^{\varepsilon}(L) = \{\lambda \in \mathbb{C} : \|R_L(\lambda)\| = \left\|(\lambda I - L)^{-1}\right\| > 1/\varepsilon\}$$

# Pseudospectrum in the self-adjoint case (Pöschl-Teller with $L_2 = 0$)

The colors correspond to $\log_{10} \varepsilon$.
The contour lines form circles centered on the eigenvalues and horizontal lines far away from the eigenvalues.

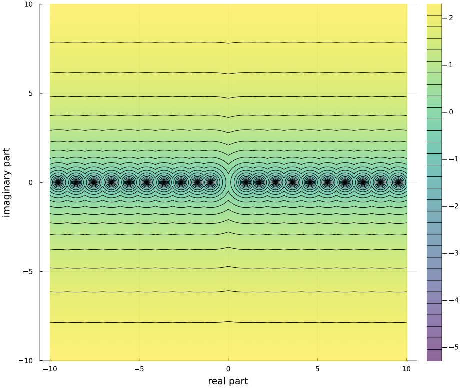



Figure: Pseudospectrum in the self adjoint case.

# Pseudospectrum (Pöschl-Teller)

The contour lines are open and the eigenvalue can migrate very far from the eigenvalues of the non perturbed operator.

Issue : The (numerical) energy pseudospectrum doesn't converge with $N$

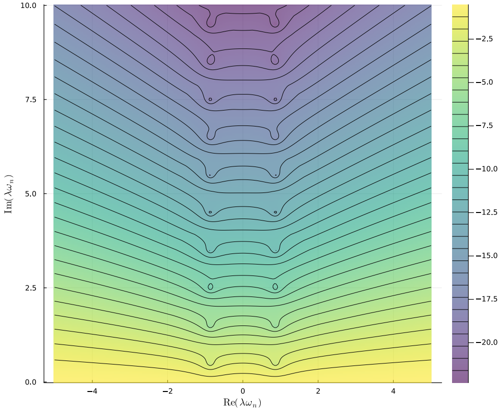



Figure: Pöschl-Teller energy pseudospectrum.

# $H^p-$pseudospectrum

## $H^p-$QNMs

$H^p$-QNMs are eigenfunctions of the $H^p$-regular operator

$$L_p \colon H^p \times H^{p-1} \to H^p \times H^{p-1}$$
$$(\phi, \psi) \mapsto L(\phi, \psi)$$

they constitute a finite set below $\mathrm{Im}(\lambda) < a + \kappa \left(p - \frac{1}{2}\right)$ with $\kappa$ the surface gravity and some constant $a$. QNMs contained in the first $p$ bands of width $\kappa$ are required to have $H^p$ regularity. We introduce a norm that make the Pöschl-Teller pseudospectrum converge in bands and increases the regularity of the QNMs in these bands.[vii]

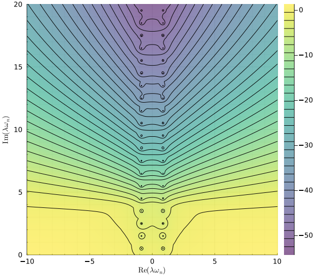



Figure: $H^4-$pseudospectrum.

$$\left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{H^p}^2 := \sum_{j=0}^{p} \left\| \begin{pmatrix} \partial_x^j \phi \\ \partial_x^j \psi \end{pmatrix} \right\|_E^2$$

[vii] Warnick;1306.5760, Boyanov,Cardoso,Destounis,Jaramillo;2312.11998

# $H^8$-pseudospectra (Pöschl-Teller case)

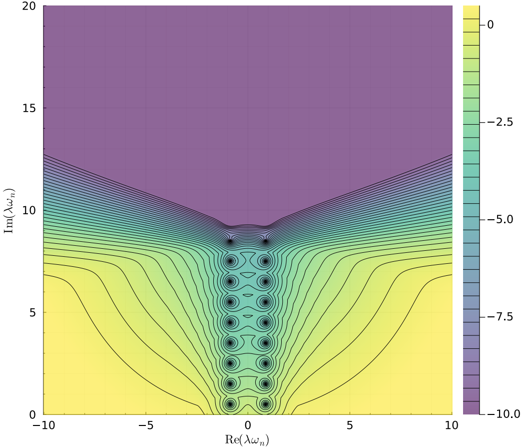



Figure: $H^8$-pseudospectrum **with fixed colorbar**.

# $H^9$-pseudospectra (Pöschl-Teller case)

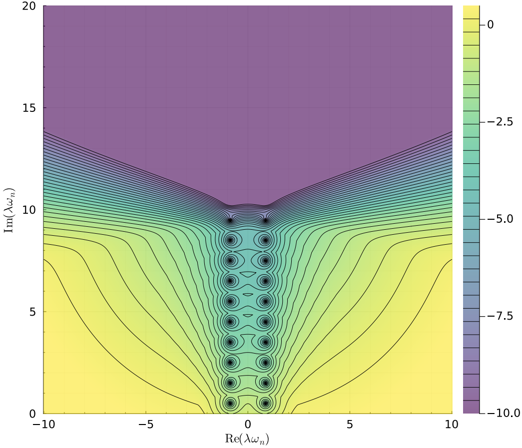



Figure: $H^9$-pseudospectrum **with fixed colorbar**.

# $H^{10}$-pseudospectra (Pöschl-Teller case)

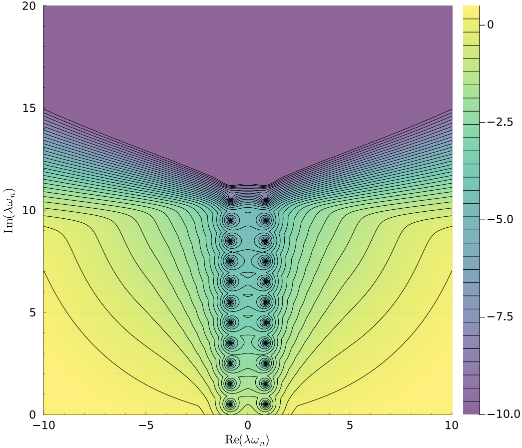



Figure: $H^{10}$-pseudospectrum **with fixed colorbar**.

# $H^{11}$-pseudospectra (Pöschl-Teller case)

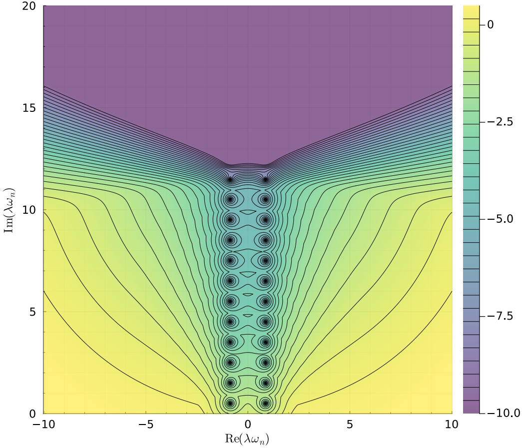



Figure: $H^{11}$-pseudospectrum **with fixed colorbar**.

# $H^{12}$-pseudospectra (Pöschl-Teller case)

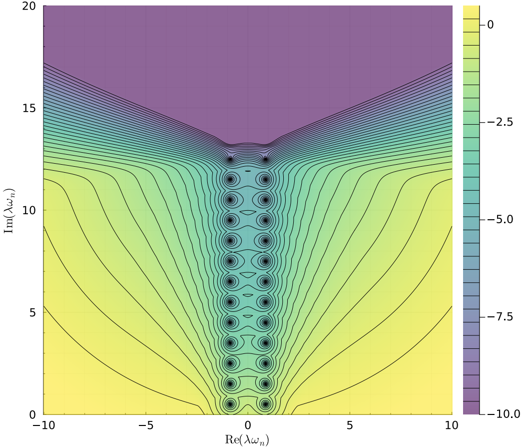



Figure: $H^{12}$-pseudospectrum **with fixed colorbar**.

# $H^{13}$-pseudospectra (Pöschl-Teller case)

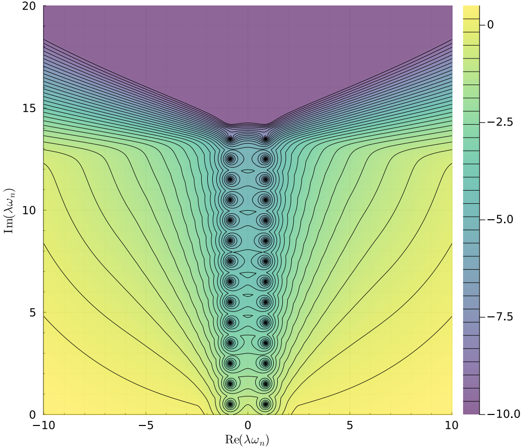



Figure: $H^{13}$-pseudospectrum **with fixed colorbar**.

# $H^{14}$-pseudospectra (Pöschl-Teller case)

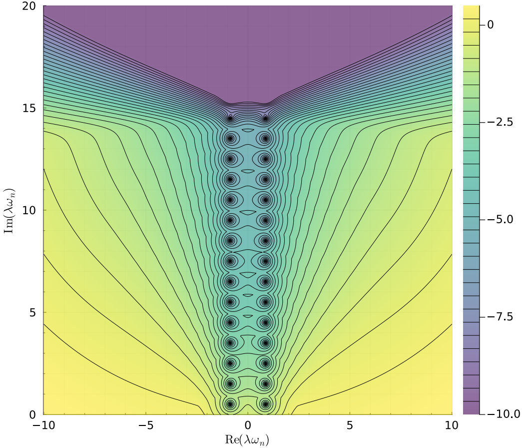



Figure: $H^{14}$-pseudospectrum **with fixed colorbar**.

# $H^{15}$-pseudospectra (Pöschl-Teller case)

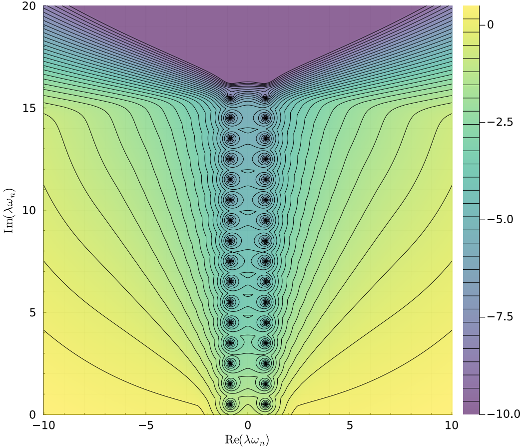



Figure: $H^{15}$-pseudospectrum **with fixed colorbar**.

# $H^{16}$-pseudospectra (Pöschl-Teller case)

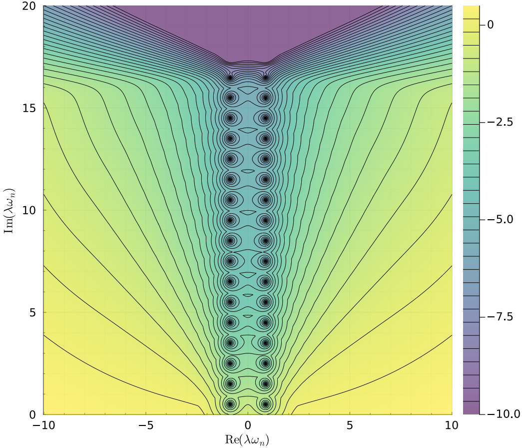



Figure: $H^{16}$-pseudospectrum **with fixed colorbar**.

# $H^{17}$-pseudospectra (Pöschl-Teller case)

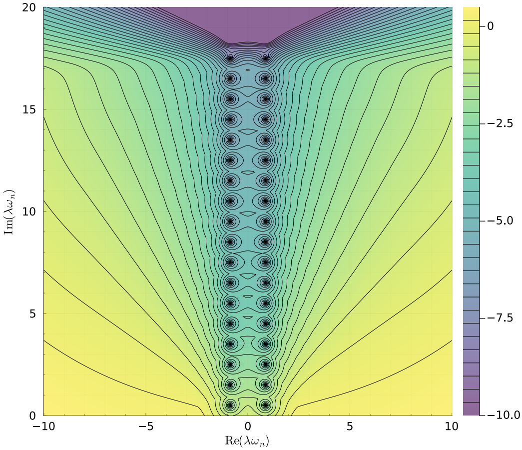



Figure: $H^{17}$-pseudospectrum **with fixed colorbar**.

# $H^{18}$-pseudospectra (Pöschl-Teller case)

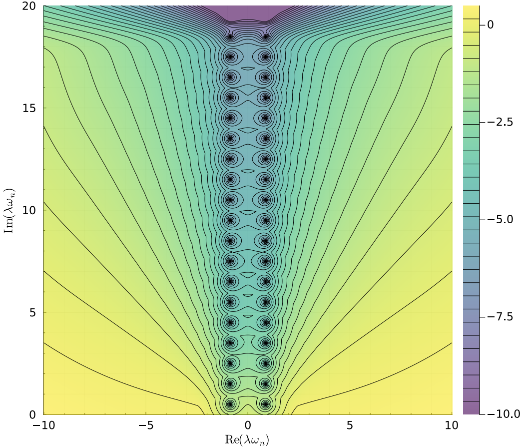



Figure: $H^{18}$-pseudospectrum **with fixed colorbar**.

# $H^{19}$-pseudospectra (Pöschl-Teller case)

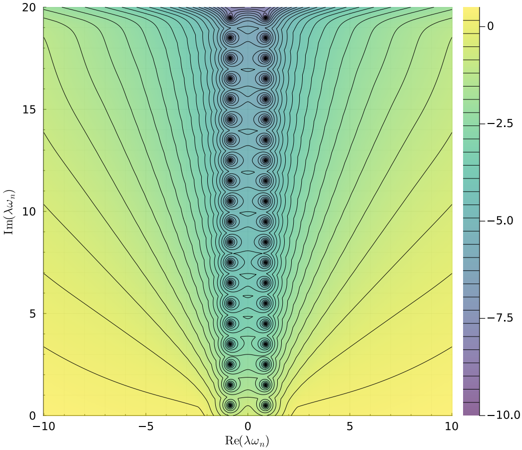



Figure: $H^{19}$-pseudospectrum **with fixed colorbar**.

# $H^{20}$-pseudospectra (Pöschl-Teller case)

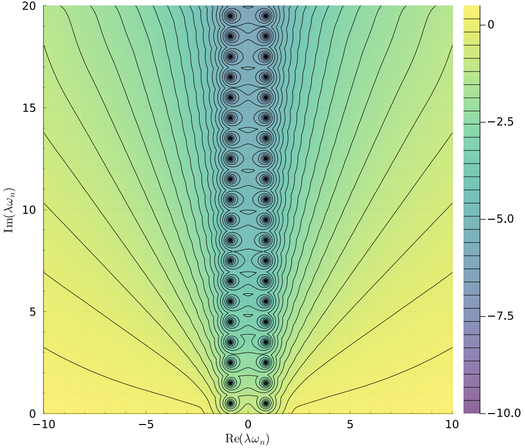
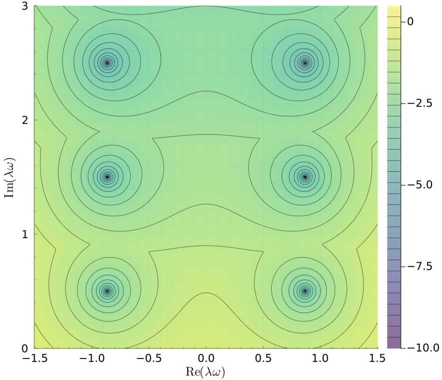



Figure: $H^{20}$-pseudospectrum **with fixed colorbar**.

# $H^{20}$-pseudospectra (Pöschl-Teller case)




(e) $H^{20}$-pseudospectrum, zoom on the first 3 modes

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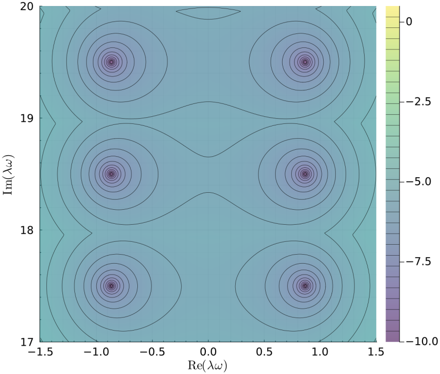


(f) $H^{20}$-pseudospectrum, zoom on the 18th to 20th modes

# Convergence test of the $H^p$-pseudospectrum

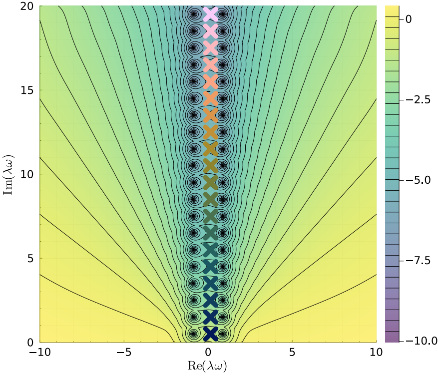


Figure: We pick some points in the complex plane

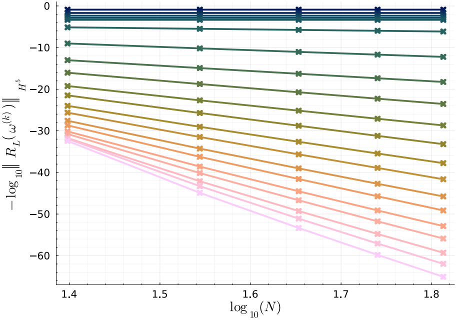


Figure: $H^5$-norm of the resolvent ($H^5$-pseudospectrum)

# Convergence test of the $H^p$-pseudospectrum

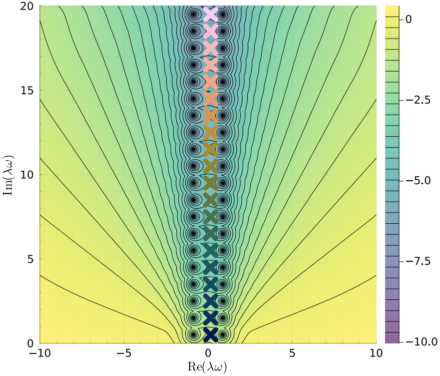


Figure: We pick some points in the complex plane

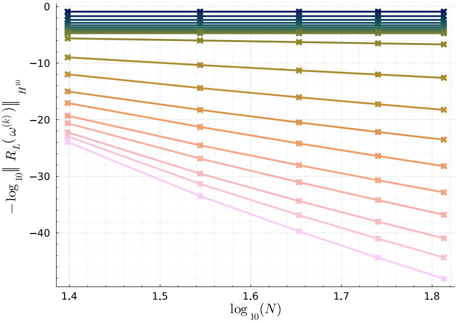


Figure: $H^{10}$-norm of the resolvent ($H^{10}$-pseudospectrum)

# Convergence test of the $H^p$-pseudospectrum

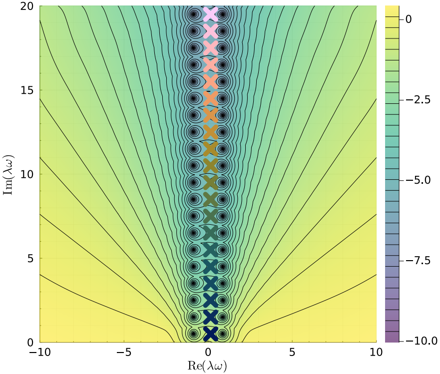


Figure: We pick some points in the complex plane

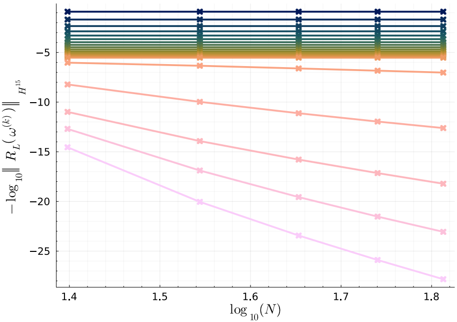


Figure: $H^{15}$-norm of the resolvent ($H^{15}$-pseudospectrum)

# Convergence test of the $H^p$-pseudospectrum

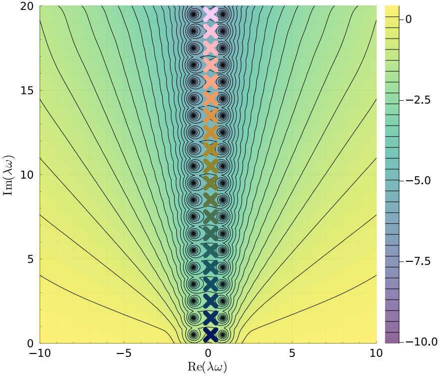


Figure: We pick some points in the complex plane

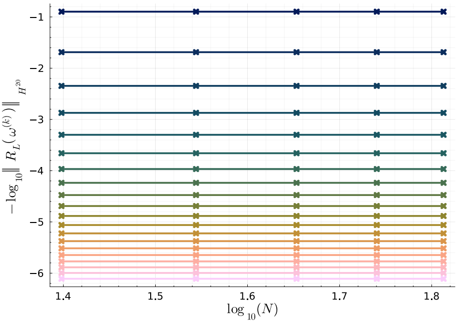


Figure: $H^{20}$-norm of the resolvent ($H^{20}$-pseudospectrum)

# Qualitative control of QNMs? Coefficients $a_n$

Notations : $\mathcal{A}_n(x) = a_n \hat{v}_n(x)$ where the $v_n$ are the normalized eigenfunctions of $L$.

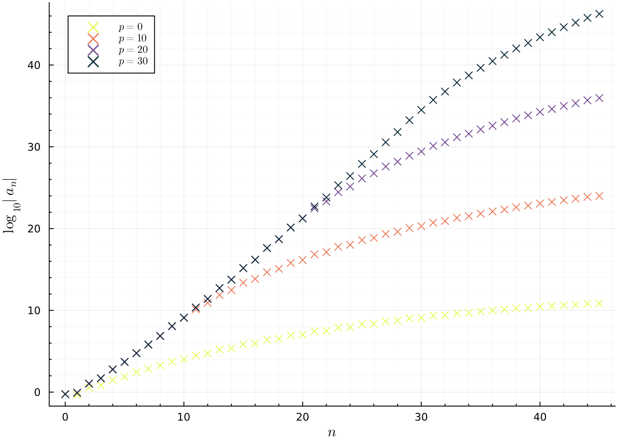


Figure: Coefficients $a_n$ for various $H^p$ norms.

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# Conclusions

- Unique expansion at null infinity
- Keldysh's approach generalizes previous QNM expansion schemes (higher dimensions, tails, ...)
- Agnostic nature of the Keldysh QNM expansion : the expansion is independant of a scalar product ("dual pairing" $\langle ., . \rangle$ notion instead)
- Polynomial tails are recovered and follow the Price law
- Role of overtones at early times of the waveforms
- $H^p$-pseudospectra converge according to Warnick's criterion
- Dynamics from the evolution operator amounts to Keldysh over all the eigenvalues of the matrix
- Error of the asymptotic expansion
- $H^p$-transients

# Conclusions

Thanks for your attention !

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# Pöschl-Teller : a toy model (part 1)

## Pöschl-Teller (1/2)

$$\left(\frac{\partial^2}{\partial \bar{t}^2} - \frac{\partial^2}{\partial \bar{x}^2} + V(\bar{x})\right)\phi = 0, \qquad V(\bar{x}) = V_0 \operatorname{sech}^2(\bar{x})$$

The following change of variable[viii] defines a compactified hyperboloidal foliation :

$$\begin{cases} \tau = \bar{t} - \ln(\cosh \bar{x}) \\ x = \tanh^{-1}(\bar{x}) \end{cases}$$

$\bar{t}, \bar{x} \in \mathbb{R}; x \in [-1, 1]$

[viii] Al Sheikh,Jaramillo,Macedo;2004.06434,Al Sheikh PhD,Bizoń,Chmaj,Mach;2002.01770

# Pöschl-Teller : a toy model (part 2)

## Pöschl-Teller (2/2)

First order reduction : $u(x,\tau) = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ with $\psi := \partial_\tau \phi$,

$$\partial_\tau \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \left( \begin{array}{c|c} 0 & 1 \\ \hline \partial_x((1-x^2)\partial_x) - V_0 & -(2x\partial_x + 1) \end{array} \right) \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Differential equation :

$$\partial_\tau u = iLu, \qquad u(x, \tau = 0) = u_0(x)$$

Spectral problem : $Lv_n = \omega_n v_n$

## Analytical Pöschl-Teller QNMs

$\phi_n(x) =$ Gegenbauer polynomials $C_n^{(i\omega_n + \frac{1}{2})}(x)$, $\qquad \omega_n = \pm\dfrac{\sqrt{3}}{2} + i\left(n + \dfrac{1}{2}\right)$

# Pöschl-Teller quasi-normal frequencies

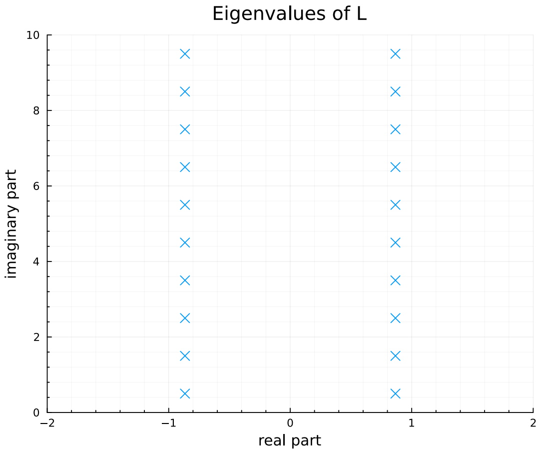



Figure: View of the Pöschl-Teller QNMs frequencies in the complex plane

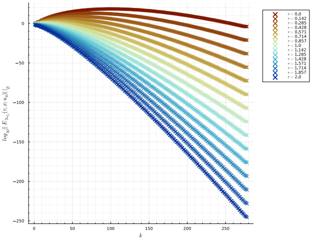
# Role of overtones : Pöschl-Teller

$$u(\tau, x) = \sum_{n=0}^{N_{\mathsf{QNM}}} \mathcal{A}_n(x) e^{i\omega_n \tau} + E_{N_{\mathsf{QNM}}}(\tau; u_0)(x), \quad \|E_{N_{\mathsf{QNM}}}(\tau; u_0)\| \leq \|u_0\| C_R(L) e^{-\operatorname{Im}\left\{\omega_{N_{\mathsf{QN}}}\right.}$$

We want to assess the convergence of the sum as a series. Does

$$\forall \varepsilon > 0, \exists M \in \mathbb{N}, \forall n > M, \|E_n(\tau; u_0)\| < \varepsilon?$$

We plot $\|E_n(\tau; u_0)\|_E$ as a function of $n$, the number of QNMs in the truncated sum.



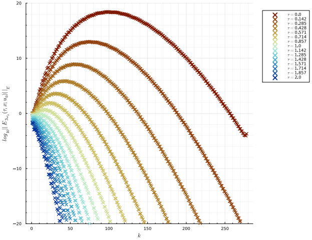
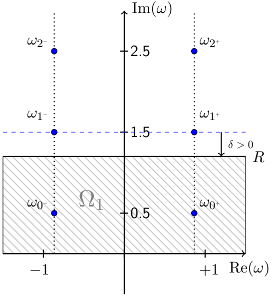




Figure: Norm of the error as we add more terms to the QNM expansion. A color corresponds to a time $\tau$.

# Asymptotic resonant expansion




## Error for $1/2 < R < 3/2$

Summing over only the fundamental mode, we have the error

$$
\begin{aligned}
E_1(x,\tau) &= u(x,\tau) - \sum_{n<\mathrm{Im}\{\omega_{1^\pm}\}} \mathcal{A}_n(x)e^{i\omega_n\tau} \\
&= u(x,\tau) - \mathcal{A}_{0^-}(x)e^{i\omega_{0^-}\tau} - \mathcal{A}_{0^+}(x)e^{i\omega_{0^+}\tau}
\end{aligned}
$$

## Bound

$$
\frac{\|E_1(\tau;u_0)\|_E}{\|u_0\|_E} \leq C_1(L)e^{-\frac{3}{2}\tau}
$$

# An approach to the convergence of the asymptotic series (motivation)

## Error bound (theorem)

$$\frac{\|E_n(\tau; u_0)\|_E}{\|u_0\|_E} \leq C_n e^{-(n+1/2)\tau}$$

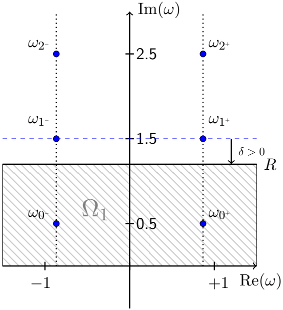
$n$ is the number of modes in the truncated expansion
$C_n$ doesn't depend on $u_0$

## Norm of a matrix $A$ with respect to the scalar product $\langle .,. \rangle_G$

$$\|A\|_G = \max_{u_0} \frac{\|Au_0\|_G}{\|u_0\|_G}$$

Can we estimate $C_n$ without fixing any particular $u_0$ ?
Idea: Get rid of $u_0$ by transforming $E_n$ into a matrix and then compute its norm.

# An approach to the convergence of the asymptotic series




$$
\begin{aligned}
\boldsymbol{E}_1(\tau)\boldsymbol{u}_0 &= \boldsymbol{u}(\tau) - \sum_{\omega_n \in \Omega_1} \mathcal{A}_n e^{i\omega_n \tau} \\
&= \boldsymbol{P}\,\mathrm{diag}\left(e^{i\omega_{0^-}\tau}, e^{i\omega_{0^+}\tau}, e^{i\omega_{1^-}\tau}, e^{i\omega_{1^+}\tau}, ...,\right)\boldsymbol{P}^{-1}\boldsymbol{u}_0 \\
&\qquad\qquad - \boldsymbol{P}\,\mathrm{diag}\left(e^{i\omega_{0^-}\tau}, e^{i\omega_{0^+}\tau}, 0, 0, ...,\right)\boldsymbol{P}^{-1}\boldsymbol{u}_0 \\
&= \boldsymbol{P}\,\mathrm{diag}\left(0, 0, e^{i\omega_{1^-}\tau}, e^{i\omega_{1^+}\tau}, ...,\right)\boldsymbol{P}^{-1}\boldsymbol{u}_0
\end{aligned}
$$

# An approach to the convergence of the asymptotic series

We compute the norm of the matrix

$$\boldsymbol{E}_1(\tau) = \boldsymbol{P}\,\mathrm{diag}\left(0, 0, e^{i\omega_{1^-}\tau}, e^{i\omega_{1^+}\tau}, ...,\right)\boldsymbol{P}^{-1}$$

We traded a depency on $u_0$ for a depency on $\tau$. We chose $\tau = 5$ for the next two figures.

The norm $\|\boldsymbol{E}_n(\tau)\|_{H^p}$ converges as the gridsize $N$ increases <u>if $n \leq p$</u>.

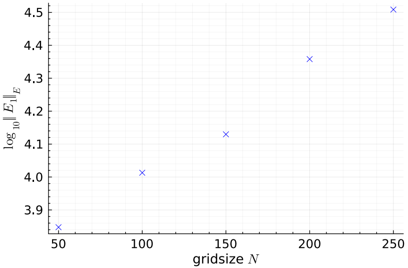



Figure: $\log_{10}$ of the energy norm of the error $E_1(\tau = 5)$

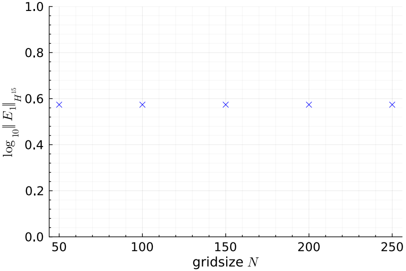



Figure: $\log_{10}$ of the $H^{15}$-norm of the error $E_1(\tau = 5)$