A new approach in classical Klein-Gordon cosmology

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Common trends in non-Hermitian Physics
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Outline

Based on: 2408.16549
Work with Nicolai Rothe

Introduction and motivation

Why are we interested in classical fields?

The semiclassical Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \langle T^{\text{ren}}_{\mu\nu} \rangle_\omega \, .$$

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Derivations of solutions:

- Cosmological spacetimes [Gottschalk, Siemssen, 2018] [Meda, Pinamonti, Siemssen, 2020]
- Einstein's static universe [Sanders, 2021]
- Black hole spacetimes: no solutions, progress on the renormalization of the quantized stress-energy tensor [Taylor, Breen, Ottewill, 2021] (and others...)

Why are we interested in classical fields?

In the moment approach [Gottschalk, Siemssen, 2018]

$$\langle T^{\mathrm{ren}}_{\mu\nu}\rangle_\omega = \langle T_{\mu\nu}\rangle_{\omega_1} + \langle T_{\mu\nu}\rangle_{\omega_2^{\mathrm{reg}}} + \Theta^{\mathrm{Had}}_{\mu\nu} + \Theta^{\mathrm{ta}}_{\mu\nu} + \Theta^{\mathrm{ren}}_{\mu\nu}(c)$$

- $\langle T_{\mu\nu}\rangle_{\omega_1}$, $\langle T_{\mu\nu}\rangle_{\omega_2^{\mathrm{reg}}}$: contributions of the one-point function and of the regularized two-point function
- $\Theta^{\mathrm{Had}}_{\mu\nu}$, $\Theta^{\mathrm{ta}}_{\mu\nu}$, $\Theta^{\mathrm{ren}}_{\mu\nu}(c)$: contributions of the Hadamard condition on the two-point function, of the trace anomaly and of the renormalization freedom, respectively.

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How do these solutions affect the full semiclassical system?

The model

FLRW spacetimes

$$\boxed{g := -dt^2 + a(t)^2 g_{\mathbb{R}^3}}$$

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The Ricci scalar

$$R[\,a\,] = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$$

The Hubble parameter and the decceleration parameter

$$H[\,a\,] : I_t \to \mathbb{R},\ t \mapsto \frac{\dot{a}(t)}{a(t)}\,, \quad q[\,a\,] : I_t \to \overline{\mathbb{R}},\ t \mapsto -\frac{a(t)\ddot{a}(t)}{\dot{a}(t)^2}$$

Any matter that solves the Einstein equation in FLRW spacetimes and γ-type solutions

$$(T^{\mu}{}_{\nu}) = \mathrm{diag}(-\varrho, p, p, p)\,, \qquad \Gamma[a] = \gamma := \frac{p}{\varrho}$$

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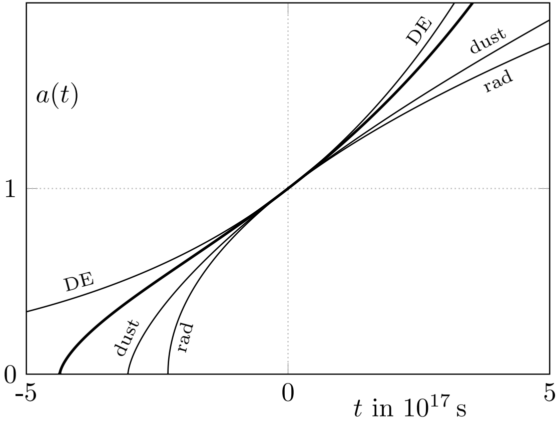
From [PLANCK, 2018]

$$\Omega_{\mathrm{rad}} = 5.38 \cdot 10^{-5}\,, \quad \Omega_{\mathrm{dust}} = 0.315\,, \quad \Omega_{\mathrm{DE}} = 0.685 \quad \text{and} \quad H_0 = 2.19 \cdot 10^{-18}\, s^{-1}$$

Constant EOS and the ΛCDM model



The classical Klein-Gordon field

The action integral

$$S_{\mathrm{KG}} = \frac{1}{2}\int d^4x\sqrt{-g}\Big(-(\nabla_\mu\phi)(\nabla^\mu\phi) + \xi R\phi^2 + m^2\phi^2\Big)\,,$$

$m \geq 0$: the mass of the field, $\xi \in \mathbb{R}$: a dimensionless coupling constant, $\xi_{\mathrm{cc}} = \frac{1}{6}$: conformal coupling, $\xi_{\mathrm{mc}} = 0$: minimal coupling

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The Klein-Gordon equation

$$\left(-\nabla^\sigma\nabla_\sigma + m^2 + \xi R\right)\phi = 0$$

The stress-energy tensor

$$\begin{aligned} T_{\mu\nu} &= (1-2\xi)(\nabla_\mu\phi)(\nabla_\nu\phi) - \tfrac{1}{2}(1-4\xi)g_{\mu\nu}(\nabla^\sigma\phi)(\nabla_\sigma\phi) - \tfrac{1}{2}g_{\mu\nu}m^2\phi^2 \\ &\quad +\xi\left(G_{\mu\nu}\phi^2 - 2\phi\nabla_\mu\nabla_\nu\phi + 2g_{\mu\nu}\phi\nabla^\sigma\nabla_\sigma\phi\right). \end{aligned}$$

Its trace

$$T^\mu{}_\mu = (6\xi-1)\left((\nabla^\mu\phi)(\nabla_\mu\phi) + \phi\nabla^\mu\nabla_\mu\phi\right) - m^2\phi^2\,,$$

$T^\mu{}_\mu = 0$ only for a massless, conformally coupled field

Setup of cosmological solutions

Three equations:

$$G_{00} + \Lambda g_{00} = \kappa T_{00} \qquad \text{Energy equation}$$

$$-R + 4\Lambda = \kappa T^{\mu}{}_{\mu} \qquad \text{Trace equation}$$

$$\left(-\nabla^{\sigma}\nabla_{\sigma} + m^2 + \xi R\right)\phi = 0 \qquad \text{Field equation}$$

Setup of cosmological solutions

Three equations:

$$0 = \dot{\phi}^2 + 12\xi \frac{\dot{a}}{a}\phi\dot{\phi} + m^2\phi^2 + 6\xi \frac{\dot{a}^2}{a^2}\phi^2 - 6\,\frac{\dot{a}^2}{a^2} + 2\Lambda$$

Energy equation

$$0 = 6\left(1 + \xi(6\xi - 1)\phi^2\right)\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - 4\Lambda + (1 - 6\xi)\dot{\phi}^2 + (6\xi - 2)m^2\phi^2$$

Trace equation

$$0 = \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m^2\phi + 6\xi\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\phi$$

Field equation

Setup of cosmological solutions

Trace and field equation:

$$\ddot{a} = -\frac{\dot{a}^2}{a} + \frac{a}{6}\left(\frac{4\Lambda + (2-6\xi)m^2\phi^2 + (6\xi-1)\dot{\phi}^2}{1+\xi(6\xi-1)\phi^2}\right)$$

$$\ddot{\phi} = -3\frac{\dot{a}}{a}\dot{\phi} - \phi\left(\frac{4\Lambda\xi + m^2 + m^2\xi\phi^2 + \xi(6\xi-1)\dot{\phi}^2}{1+\xi(6\xi-1)\phi^2}\right)$$

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Critical value of ϕ

$$1 + \xi(6\xi-1)\phi^2_{\text{crit}}(\xi) = 0$$

Framework of constraints

$$\ddot{a} = -\frac{\dot{a}^2}{a} + \frac{a}{6}\left(\frac{4\Lambda + (2-6\xi)m^2\phi^2 + (6\xi-1)\dot{\phi}^2}{1+\xi(6\xi-1)\phi^2}\right)$$

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Idea

Use the energy equation as one constraint and the value of $q_0 = -\ddot{a}(0)/\dot{a}(0)^2$ as another. We have $q_{\Lambda CDM} = -0.538$.

Constraints from the energy equation

We consider solutions to the energy equation which are conic sections in $(\phi, \dot{\phi}, H)$: $0 = f(\phi, \dot{\phi}, H)$
For $0 < \xi < 1/6$: ellipse

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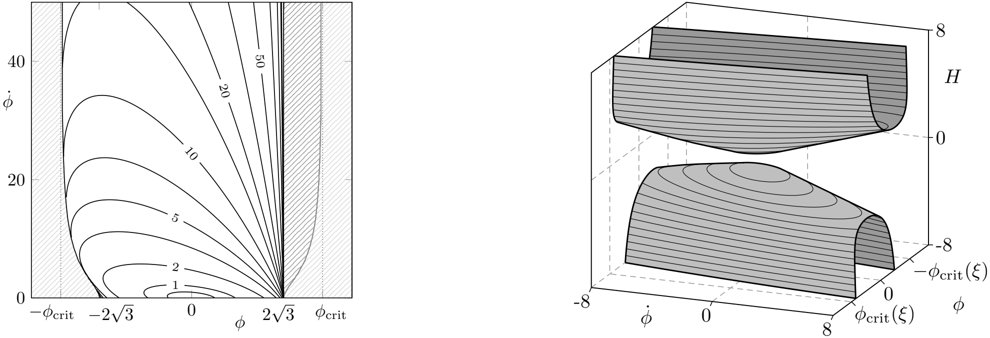

$\Lambda = m = 1$ and $\xi = 1/12$

Constraints from a prescribed initial deceleration parameter

Parameter space (Λ, m, ξ) with a fixed q_0:

1. No solutions (no real values of $\phi(0)$ and $\dot{\phi}(0)$)
2. Two solutions (one cosmology)
3. Four solutions (two inequivalent cosmologies)

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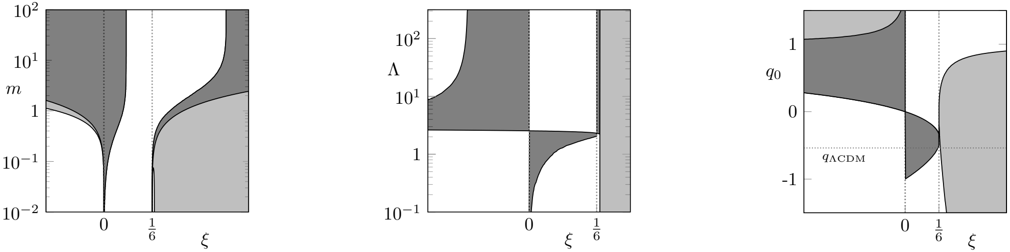

$\Lambda = 0,\ q_0 = q_{\Lambda CDM}$ $m = 0,\ q_0 = q_{\Lambda CDM}$ $\Lambda = 2,\ m = 0$

Early and late time behavior

Types of singularities

Big Bang singularity

Let $M = (t^{s}, T) \times \mathbb{R}^3$ be a flat FLRW spacetime defined by the scale factor $a : (t^{s}, T) \to (0, \infty)$ and let $\gamma \in \mathbb{R}$. M is said to have a γ-type Big Bang singularity at $t = t^{s}$ if

$$a(t) \to 0\,, \quad \dot{a}(t) \to \infty \qquad \text{and} \qquad \Gamma[\,a\,](t) \to \gamma$$

as $t \to t^{s}$.

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as $t \to t^s$.

Small Bang singularity

Let $M = (t^s, T) \times \mathbb{R}^3$ be a flat FLRW spacetime defined by the scale factor $a : (t^s, T) \to (a_{\mathrm{SB}}, \infty)$. M is said to have a Small Bang singularity in $t = t^s$ if

$$a(t) \to a_{\mathrm{SB}} \qquad \text{and} \qquad \dot{a}(t) \to \infty$$

as $t \to \infty$, for some $a_{\mathrm{SB}} > 0$.

γ-type solutions

For $\xi > 1/6$: $a(t) \to 0$ and $\Gamma[a](t) \to \gamma$ as $t \to t^s$.

Three types of early time behavior:

- Type I: The pure radiation expansion (Big Bang singularity)
- Type II: The γ-type Big Bang singularity
- Type III: The scale factor asymptotically approaches zero.

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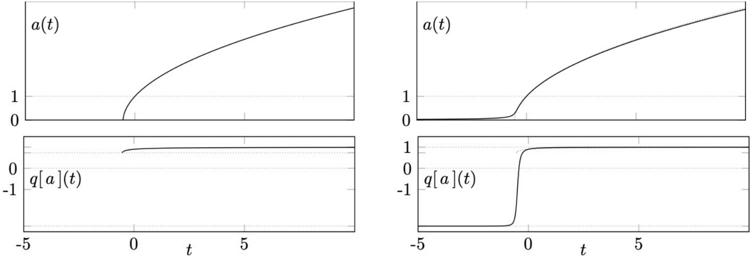

Type II Type III

$\Lambda = m = 0$, $\xi = \frac{1}{4}$ and $q_0 = 0.9$

Inflation

Definition

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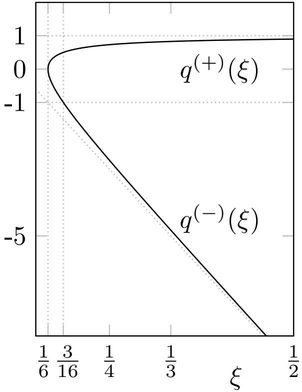
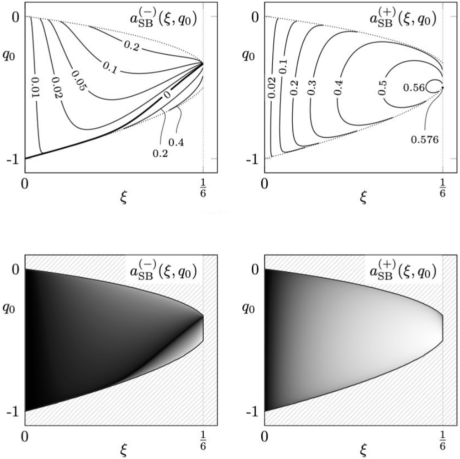


$m = 0.2$ and $\Lambda = 0.001$

- Distinguished value of $\xi = 3/16$
- Inflationary period without other assumptions (e.g. slow roll conditions)
- Further numerical study of this period to determine if it lasts long enough to solve the cosmological problems or generate density fluctuations

Small Bang solutions

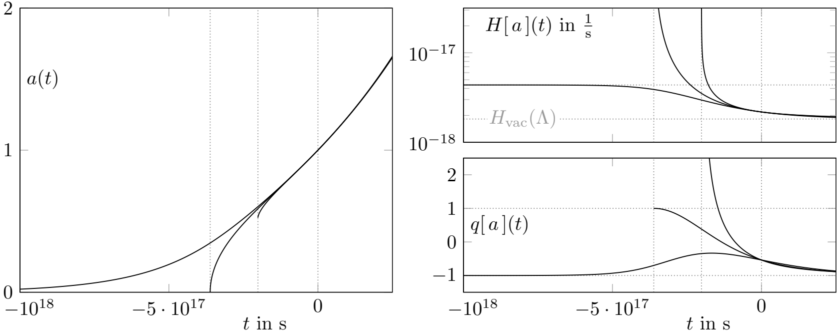
Small Bang solutions are generic for $0 < \xi < 1/6$



Late times evolution

Numerical evidence

For all solutions with $\xi \geq 0$ and $\Lambda > 0$ we observed a de Sitter late-time expansion with $q[\,a\,](t) \to -1$ and $H[\,a\,](t) \to H_{\text{vac}}(\Lambda)$ as $t \to \infty$



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- In a new approach to classical cosmology we showed the allowed range of parameters to have real solutions
- At early times we observed Big Bangs, Small Bangs and an early inflationary period for different values of the coupling constant
- At late times all solutions with positive cosmological constant approach a de Sitter universe